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# Effect of slight stratification on the nonlinear spatial evolution of a weakly unstable wave in a free shear layer

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We examine the spatial evolution of an instability wave excited by an external source in a free, nearly non-dissipative, stably stratified shear flow with a small Richardson number $Ri \ll 1$. It turns out that at the nonlinear stage of evolution even so small a stratification modifies greatly the evolution behaviour compared with the case of a homogeneous flow which was studied in detail by Goldstein & Hultgren (1988).

We have investigated (analytically and numerically) different stages of evolution corresponding to different critical layer regimes, and determined the formation conditions and structure of a quasi-steady nonlinear critical layer.

It is shown that the stratification influence upon the nonlinear evolution is governed by the parameter $(Pr-1)Ri/\gamma_L^2$, where $Pr$ is the Prandtl number and $\gamma_L$ is the wave's linear growth rate (which is a measure of supercriticality), and this effect is important only when $\gamma_L < Ri^{1/2}$, $Pr \neq 1$. The character of this influence radically depends on the sign of $(Pr-1)$. Thus, when $Pr < 1$ the amplitude in the course of the evolution varies in a limited range and either reaches saturation, when the supercriticality is small enough or, at higher supercriticality, performs quasi-periodic oscillations, whose structure becomes increasingly complicated with increasing $\gamma_L$. When $Pr > 1$ stratification leads to the appearance of new evolutionary stages, namely the stage of explosive growth in the unsteady critical layer regime, and the stage of essentially unsteady evolution in the nonlinear critical layer regime, and to a modification of the power-law growth in the regime of a quasi-steady nonlinear critical layer.

## 1. Introduction

In this paper we pursue our previous investigation of the dependence of the character of the weakly nonlinear evolution of unstable disturbances in shear flows with large Reynolds numbers on the behaviour of a non-dissipative neutral mode near a critical level $y = y_c$ where the flow velocity $v_x = u(y)$ coincides with the wave's phase velocity $c$, $u(y_c) = c$, and wave–flow resonance interaction occurs.

It is well known (e.g. Drazin & Reid 1981) that the point $y = y_c$ is singular for steady linearized and inviscid hydrodynamic equations describing a neutral mode and in a general case the neutral mode itself is also singular† at $y = y_c$, namely it has

† Broadly speaking, a disturbance is a multicomponent one (different velocity components, density, etc.). Some of the components can split off and may not affect the dynamics of the others, which will be referred to as essential. The mode will be described by a vector-function consisting only of essential disturbance components. The neutral mode is singular if at least one of its components is singular.

a pole or a branch point. In some instances, however (two-dimensional disturbances in a homogeneous incompressible medium, for example), the neutral mode is regular when $y = y_c$.

From a physical point of view, the singularity of the equations means that in a neighbourhood of $y = y_c$, a so-called critical layer (CL), one cannot neglect simultaneously the dissipation, the disturbance evolution and the nonlinearity. Each of these factors individually eliminates the singularity by modifying, in its own way, the equations when $|y - y_c| < l_i$, $i = v, t, N$, where

$$l_v = v^{1/3}, \quad l_t = |A|^{-1}\mathrm{d}|A|/\mathrm{d}s, \quad l_N = A^p \tag{1.1}$$

are the viscous, the unsteady (evolution) and the nonlinear scales, respectively. Here $v$ is the reciprocal of the Reynolds number, $A$ is a complex amplitude of the disturbance, $s$ is the evolution variable (the time or downstream coordinate), and $p$ is a parameter depending on the Richardson number $Ri$ and varying from 1/2 in a homogeneous medium to 2/3 in a medium with sufficiently large stratification ($Ri \equiv Ri(y_c) \geqslant 1/4$).

Depending on which of these three scales (1.1) is the greatest, the corresponding term (viscous, unsteady or nonlinear) appears in the differential operator of the CL equations and the two others appear on the right-hand sides as disturbing terms. In accordance with the structure of the CL equations three CL regimes are conveniently distinguished: viscous (dissipative), unsteady and nonlinear. In the viscous and unsteady CL regimes the left-hand sides of the CL equations are linear and these CL regimes are also called linear. On the other hand, the nonlinearity in the evolution equation for the amplitude of the disturbance is due to the generation of harmonics and their nonlinear interactions which occur largely inside the CL and are governed by the corresponding CL equations. Therefore, to each CL regime (even to the *linear* one) there corresponds quite a definite type (set of types) of *nonlinear* evolution equation (NEE) and, accordingly, a definite character of evolutionary behaviour of the disturbance. Specifically, the Landau–Hopf evolution scenario (the creation of a limiting cycle) is realized only in the viscous CL regime; the nonlinear evolution in the unsteady CL regime is described by a NEE with a non-local nonlinearity, the first example of which was obtained by Hickernell (1984), and proceeds in an explosive way (Churilov & Shukhman 1988)

$$|A| \propto (s_0 - s)^{-\alpha}, \quad \alpha > 0; \tag{1.2}$$

while the nonlinear CL regime involves a nonlinear reduction of growth rate, and the disturbance evolution is generally 'slow', both in terms of amplitude growth rate (e.g. Huerre & Scott 1980; Churilov & Shukhman 1987*a*; Goldstein & Hultgren 1988)

$$|A| \propto s^{\beta}, \quad \beta > 0, \tag{1.3}$$

and in terms of the quasi-steady character of a flow inside the CL.

In the early stage of instability development the disturbance grows, in accordance with linear theory, exponentially with growth rate $\gamma_L$ determined by supercriticality, and the CL is viscous ($\gamma_L < v^{1/3}$) or unsteady ($\gamma_L > v^{1/3}$). Such a growth continues until a threshold of nonlinearity is attained, i.e. an amplitude level when nonlinear terms in the NEE become of the same order of magnitude as linear ones. The nonlinear stage of evolution starts here, and the growth in amplitude in this case can speed up, slow down, cease (the Landau–Hopf scenario) or even turn back (e.g. Wu, Lee & Cowley 1993). At the same time the CL regime is not determined for all time. As is evident from (1.1), only $l_v$ is determined solely by flow parameters, while $l_t$ and $l_N$ may vary during the course of the evolution; therefore, transitions from one CL regime to another are possible. The attainment of a threshold of nonlinearity and transitions from one CL regime to another are events that mark boundaries between different stages of disturbance evolution (each of which has its own type of evolution equation and its own evolution character) and are therefore the key points of the evolution scenario.

Churilov & Shukhman (1992) have ascertained an intimate relationship between the evolution scenario and the neutral mode behaviour near a critical level and showed that there exist two main nonlinear evolution scenarios for unstable disturbances.

In a general case of a singular neutral mode the threshold of nonlinearity is low, such that for any supercriticality ($\gamma_L \ll 1$) the nonlinear stage of disturbance development has already started in the linear (viscous or unsteady, respectively) CL regime. A nonlinear development in the unsteady CL regime means an explosive growth of the form (1.2) and in a general way, the unsteady scale $l_t$ grows with amplitude so fast that it always remains larger than $l_N$, i.e. there is no transition (up to $|A| = O(1)$ where weakly nonlinear theory is no longer valid) to the nonlinear CL regime with an inherent 'slow' law of growth of the form (1.3). The amplitude of disturbances with a smaller supercriticality ($\gamma_L < \nu^{1/3}$), that 'start' in the viscous CL regime, varies in a limited range in the same CL regime when the nonlinearity has a stabilizing character or grows explosively with the subsequent transition to the unsteady CL regime when it has a destabilizing character. All disturbances with a singular neutral mode studied so far, namely two-dimensional in stratified (Churilov & Shukhman 1987*b*, 1988) or compressible (Goldstein & Leib 1989; Shukhman 1991; Leib 1991) flows and three-dimensional in a homogeneous medium (Goldstein & Choi 1989; Wu *et al.* 1993; Wu 1993*a,b*; Churilov & Shukhman 1994; Wu & Cowley 1995), evolve according to this ('fast') scenario.

In those degenerate cases when the neutral mode is regular (two-dimensional disturbances in a homogeneous incompressible medium), the evolution is 'slow': the nonlinearity (when $\gamma_L > \nu$) turns out to be non-competitive in the viscous and unsteady CL regimes, and the exponential growth ends in the transition to the nonlinear CL regime when $A \sim \max(\nu^{2/3}, \gamma_L^2)$, and the subsequent growth proceeds according to (1.3) with $\beta = 2/3$ (Huerre & Scott 1980; Churilov & Shukhman 1987*a*; Shukhman 1989; Hultgren 1992). This scenario with full details of the transition to the nonlinear CL regime at different values of supercriticality was investigated most thoroughly by Goldstein & Hultgren (1988, hereinafter referred to as G&H) by considering an example of a free mixing layer.

These are the two main weakly nonlinear scenarios of instability development in shear flows: 'fast' ending in an explosive growth of amplitude in the unsteady CL regime up to the boundary of the validity range of the theory ($A = O(1)$), and 'slow' with the transition to the nonlinear CL regime. Note that even a linear analysis (singularity or regularity of the neutral mode) shows which scenario is realized, and complicated and unwieldy nonlinear calculations make it possible only to specify the details (the level of the threshold of nonlinearity as a function of $\gamma_L$, the value of the index $\alpha$ in (1.2) (or $\beta$ in (1.3)), etc.).

The difference between these scenarios, as is easy to see, is very large, and the question naturally arises: what scenarios are between them, in intermediate cases? To answer this question, it seems necessary to find and study the cases when the neutral mode is 'weakly singular', which can manifest itself, for example, as an 'almost splitting off' of the singular part of the disturbance and its weak influence upon the regular part. One such problem was considered by Churilov & Shukhman (1995, hereinafter referred to as C&S95), and another will be treated in the present paper.

C&S95 was devoted to the study of the streamwise development of 'weakly three-dimensional' disturbances in the form of a monochromatic running wave in a homogeneous incompressible medium. In this problem the intensity of the effect of the singular component on the regular component is governed by the parameter $\delta = (k_z/k_x)^2$, where $k_x$ and $k_z$ are the wave vector components, and as $\delta$ varies from $\delta = 0$ to $\delta = O(1)$, the transition from two-dimensional disturbances (G&H, 'slow' evolution) to three-dimensional ones (Churilov & Shukhman 1994, 'fast' evolution) occurs. The explosive growth stage inherent in fast evolution scenarios appears only at a sufficiently large $\delta$ in a limited range of variation in supercriticality ($\nu^{1/3} < \gamma_L < \delta$). When $\delta \ll 1$ this stage is only intermediate asymptotics: because of the smallness of $\delta$ the unsteady scale $l_t$ grows insufficiently fast, so that when $A = O(\delta^2)$ it is overtaken by $l_N$, and the transition to the nonlinear CL regime occurs, i.e. to the slow evolution accoridng to (1.3).

Physically, the transition to the nonlinear CL regime means that the movement of liquid particles trapped by the wave becomes the fastest process in the CL. In all cases studied heretofore, the mixing of liquid particles inside the cat's eyes caused by this movement 'grinds' down the vorticity gradient which is the cause of an instability of shear flows (for a detailed discussion see, for example, C&S95, pp. 60–61). As a result the disturbance evolution is decelerated abruptly and becomes quasi-steady, no matter how violent (even explosive, see C&S95) it was before.

We devoted a separate paper (Churilov & Shukhman 1996, hereinafter referred to as C&S96) to the study of the quasi-steady evolution of unstable disturbances and the influence of different factors upon it, including a very weak ($Ri < \nu^{2/3}$) stratification. By analysing the nonlinear evolution equations obtained for this problem, we have seen that at the Prandtl number $Pr < 1$, when density disturbances diffuse faster than velocity disturbances, there arises a factor which intervenes in the mixing process of liquid particles trapped by the wave and, at a stronger stratification ($Ri > \nu^{2/3}$), can hamper the establishment of the quasi-steady evolution of disturbances after transition into the nonlinear CL regime.

In this paper we investigate the spatial evolution of two-dimensional unstable disturbances in a shear flow with moderately weak stratification ($\nu^{2/3} \ll Ri \ll 1$). It has several aspects of interest.

In the first place, a stratification (let it be small) is inevitably present in the real medium and often plays an important role (e.g. Turner 1973; Tritton & Davies 1981). Therefore, ascertaining the validity range of models with a homogeneous medium for describing various phenomena (e.g. instability development which leads to turbulence in shear flows) is of much current interest and importance.

Secondly, the neutral mode is also weakly singular here (the governing parameter $\delta = Ri^{1/2} \equiv [Ri(y_c)]^{1/2}$), but the interaction mechanism for the singular (density perturbation) and the regular (velocity perturbation) components is totally different from the case of weakly three-dimensional disturbances. This latter factor, as the analysis shows, is no bar to the transition to the nonlinear CL regime, but it leads to a previously unknown type of evolution behaviour: an unsteady development of a disturbance in the nonlinear CL regime in a reasonably wide (when $Ri \gg \nu^{2/3}$) range of amplitudes and growth rates

$$(\nu Ri)^{2/5} < |A| < Ri\ , \quad |A| < \gamma_L^2\ , \tag{1.4}$$

which is possible to investigate only by numerical methods.

It turns out that if the Prandtl number $Pr > 1$ the disturbance ultimately 'overcomes' this gap and continues to grow further in a quasi-steady way according to (1.3). For this quasi-steady stage we obtained an analytic (in addition to numerical) solution taking into account also corrections for a weak unsteadiness. On the one hand, it serves as a check of the numerical solution and, on the other, provides a significant amount of additional information about the nonlinear CL properties in a stratified medium. In particular, the analytic solution indicates that the quasi-steady evolution of sufficiently supercritical disturbances ($\gamma_L > \nu^{1/3}$) is impossible in the nonlinear CL regime when $Pr < 1$ and stratification is strong enough ($Ri > \gamma_L^2$). Indeed, the numerical analysis shows that in this case the disturbance gets trapped in the gap (1.4), and its amplitude oscillates in a limited range.

Thirdly, the very fact of nonlinear CL formation in a (weakly) stratified medium as a result of the instability development is remarkable and interesting since it is well known (Churilov & Shukhman 1988) that with a finite stratification ($Ri \approx 1/4$) the evolution is fast and the nonlinear CL regime is not realized. It might be well to point out that solutions for a *steady* nonlinear CL were constructed relatively long ago. Kelly & Maslowe (1970) pioneered a treatment of this problem as applied to the case of a weak stratification, and Haberman (1973) made a correct matching of the solution across the cat's eye boundary based on the same principles as in his earlier work (Haberman 1972) devoted to a nonlinear CL in a homogeneous medium.

This paper is organized as follows. In § 2 we give the statement of the problem, the scaling and a brief derivation of the basic equations. The evolution in the viscous and unsteady CL regimes is considered in § 3. In § 4 an analytic study is made of the nonlinear quasi-steady CL regime and its boundaries are found. The procedure and results of a numerical calculation are discussed in § 5. Finally, § 6 is devoted to a discussion of results obtained. The Appendix includes the derivation of nonlinear evolution equations for the quasi-steady nonlinear CL regime with due regard for unsteadiness in outer diffusion layers.

## 2. Problem statement and basic equations

Consider a plane-parallel shear flow of a stratified ($\rho = \rho_0(y)$) incompressible fluid along the $x$-axis, $v_x = u(y) > 0$, with a monotonic velocity profile (it is assumed that $u_c'(y) > 0$ because the resulting evolution equations acutally involve only $u_c'^2$) and a large Reynolds number. Stratification is taken to be stable, and the density variation scale is considered to be of the same order or larger than the velocity variation scale $d$. The Richardson number is small everywhere, $Ri = -g\rho_0'/(\rho_0 u'^2) \ll 1/4$, and stratification is taken into account as a correction. Such a flow has a wide spectrum of unstable modes and applying the weakly nonlinear theory (which treats a nearly stable wave rather than the fastest growing one) needs justification. There are several ways to do this (e.g. G&H; Hultgren 1992; Churilov & Shukhman 1994). We will assume that a perturbation is produced by an external source which sets an appropriate frequence $\omega$ somewhat smaller than a critical frequency $\omega_{cr}$ (i.e. the neutral mode frequency): $\omega_{cr} - \omega \ll \omega_{cr}$. We suppose also that the amplitude of a perturbation generated in such a way is large enough to neglect more unstable disturbances that could arise from (very-low-amplitude) noise and small enough to obey the linear evolution equation near the source. The selected perturbation increases downstream, and at some distance its evolution becomes nonlinear; this is the process we are studying. A typical stability diagram and the mode selected are shown in figure 1.

Since the stratification is weak the critical level $y = y_c$ coincides, in the first approximation, with the inflection point $u_c'' \equiv u''(y_c) = 0$. By taking $|\rho_c' d|$, $d$ and $2d/\Delta u$, to be the units of density, length and time, respectively, where $\rho_c' \equiv \rho_0'(y_c)$ and

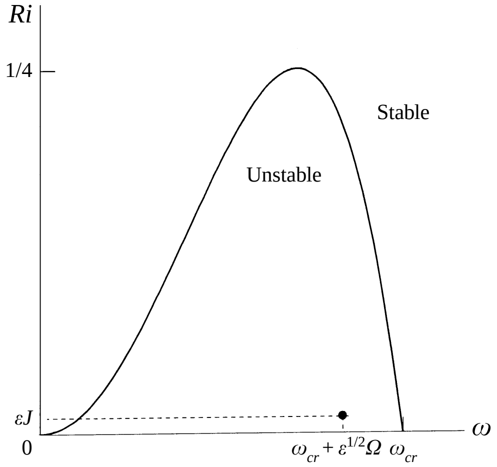


FIGURE 1. A typical curve of neutral stability on the $(Ri, \omega)$-plane. The black circle corresponds to the selected weakly unstable mode. (The neutral curve for Drazin's (1958) model is shown for which $u(y) = \bar{U} + \tanh(y)$, $r(y) = 1$, $k = \omega/\bar{U}$, $Ri = k^2(1-k^2)$, $c = \bar{U}$, $\omega_{cr} = \bar{U}$.)

$\Delta u$ is the velocity difference across the shear layer, we write the input equations in the Boussinesq approximation in a dimensionless form:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\Delta\psi + u\frac{\partial}{\partial x}\Delta\psi - u''\frac{\partial\psi}{\partial x} - N^2\frac{\partial\rho}{\partial x} + \{\Delta\psi, \psi\} &= \nu\Delta^2\psi, \\ \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + r(y)\frac{\partial\psi}{\partial x} + \{\rho, \psi\} &= \frac{\nu}{Pr}\Delta\rho, \end{aligned}\right\} \tag{2.1}$$

where $\psi$ and $\rho$ are perturbations of streamfunction and density respectively, $N^2 = -g\rho_c'/\rho_0 \approx$ const, $r(y) = \rho_0'(y)/\rho_c'$, $Pr = \nu/\kappa =$ const $= O(1)$ is the Prandtl number, $\nu$ is kinematic viscosity (inverse Reynolds number), $\kappa$ is the diffusion coefficient for density changes, $\{a, b\} = a_x b_y - a_y b_x$ and $\Delta$ denotes the Laplacian.

It is easy to see that the neutral mode does indeed consists of regular ($\psi$) and singular ($\rho$) – at a critical level – components split when $Ri = 0$ and weakly coupled when $Ri \ll 1/4$. Therefore, the solution outside the CL (i.e. of the outer problem) in the first approximation is

$$\psi = 2\varepsilon B g(y)\cos\theta, \quad \rho = -\frac{r(y)\psi}{u-c}; \quad \theta = kx - \omega t + \Theta, \tag{2.2}$$

where $g(y)$ is the eigenfunction of the neutral mode of the Rayleigh equation

$$g'' - \left(\frac{u''}{u-c} + k^2\right) g = 0, \quad g \to 0 \text{ as } y \to \pm\infty,$$

$$g = 1 + \alpha_1(y-y_c) + \alpha_2(y-y_c)^2 + \cdots, \quad y \to y_c; \quad \alpha_2 = \left(u_c'''/u_c' + k^2\right)/2,$$

and $\Theta$ and $k = \omega_{cr}/c$ are its phase and wavenumber respectively and $\varepsilon \ll 1$ is a small parameter characterizing the disturbance amplitude; the coefficient $\alpha_1$ is to be determined by solving the boundary-value problem for $g(y)$.

To be able, in terms of a unified approach, to study the three CL regimes: viscous, unsteady and nonlinear, it is assumed that the respective scales

$$l_\nu \sim \nu^{1/3}, \quad l_t \sim B^{-1}\mathrm{d}B/\mathrm{d}x, \quad l_N \sim \varepsilon^{1/2}$$

are of the same order of magnitude, and we introduce

$$\xi = \varepsilon^{1/2}x, \quad y - y_c = \varepsilon^{1/2}Y, \quad \omega = \omega_{cr} + \varepsilon^{1/2}\Omega, \quad \nu = \eta\varepsilon^{3/2} . \tag{2.3}$$

The amplitude $B$ and phase $\Theta$ depend on the evolution variable $\xi$.

Following the lead of a number of earlier articles, we will employ matched asymptotic expansions for constructing the solution of (2.1): first we seek the solutions outside and inside the CL in the form of power series in $\varepsilon$, and they are then matched in each order. Matching yields nonlinear evolution equations as compatibility conditions. Let us take a brief look at some of the main points only.

In the outer solution it is necessary to take into account, in addition to the neutral mode (2.2), corrections for the supercriticality and evolution ($O\left(\varepsilon^{3/2}\right)$) and corrections for stratification ($O(Ri\,\varepsilon)$). With the chosen scaling (2.3), we allow for any type of CL, including the nonlinear one, so that inside it all disturbance harmonics (except for the fundamental one) must be of the same order of magnitude. Because of degeneracy of this problem ($u_c'' = 0$) this order is $O(\varepsilon^2)$ (rather than $O(\varepsilon^{3/2})$ as in the case $u_c'' \neq 0$ when even $g(y)$ is singular, see Kelly & Maslowe 1970; Haberman 1973). Therefore for contributions due to stratification to be competitive, we must put $Ri = O(\varepsilon)$ (cf. scaling $Ri = O(\varepsilon^{1/2})$ in cited papers). We define

$$Ri\,(y_c) = \varepsilon J, \quad J = O(1) . \tag{2.4}$$

It is pertinent to note that, with such a scaling, stratification modifies the linear properties of disturbances so little (a displacement of the stability boundary $\Delta\omega_{cr} = O(Ri) = O(\varepsilon)$, while the supercriticality is $O\left(\varepsilon^{1/2}\right)$) that this may be neglected.

The inner (as $y \to y_c$) asymptotic expansion of the outer solution has the form

$$\psi = 2\varepsilon B_C + 2\varepsilon^{3/2}\alpha_1 B_C Y + \varepsilon^2\left[\left(\frac{u_c'''}{u_c'} + k^2\right)B_C Y^2 + 2\frac{u_c'''}{k u_c'^2}(\Omega B_C - c\dot{B}_S)Y\ln|\varepsilon^{1/2}Y| \right.$$
$$\left. +2JB_C\ln|\varepsilon^{1/2}Y| + a^{\pm} + b^{\pm}Y\right] + \cdots, \tag{2.5}$$

$$\rho = -2\varepsilon^{1/2}(B_C/u_c')Y^{-1} + \cdots, \quad B_C \equiv B\cos\theta, \quad B_S \equiv B\sin\theta; \quad \dot{B}_{C,S} \equiv \mathrm{d}(B_{C,S})/\mathrm{d}\xi .$$

The coefficients $b^{\pm}$ satisfy the modified solvability condition (MSC)

$$b^+ - b^- = -(2/k)(\Omega B_C - c\dot{B}_S)I_1 - 4kI_2\dot{B}_S; \quad I_1 = \fint_{-\infty}^{\infty}\mathrm{d}y\frac{u''g^2}{(u-c)^2}, \quad I_2 = \int_{-\infty}^{\infty}g^2\mathrm{d}y, \tag{2.6}$$

where $\fint$ stands for a Cauchy principal value. Matching to the solution inside the CL makes it possible to determine $a^{\pm}$ and $b^{\pm}$ which – upon substitition into MSC (2.6) – gives the nonlinear evolution equation.

The inner solution has to be matched to (2.5) and therefore it is sought in the form of an expansion in $\varepsilon^{1/2}$:

$$\psi = \varepsilon\left(\Psi^{(1)} + \varepsilon^{1/2}\Psi^{(2)} + \varepsilon\Psi^{(3)} + \cdots\right), \quad \rho = \varepsilon^{1/2}P^{(1)} + \cdots \quad . \tag{2.7}$$

The first two iterations of $\Psi$, in view of the matching to (2.5), are calculated in a straightforward way: $\Psi^{(1)} = 2B\cos\theta$, $\Psi^{(2)} = 2\alpha_1 BY\cos\theta$. The full $O(\varepsilon)$ streamfunction $\psi$ is $u_c'Y^2/2 + 2B\cos\theta$, which means that in the leading order streamlines have the characteristic form of cat's eyes.

The third iteration for $\psi$ and the first for $\rho$ give the desired equations. Introducing $\zeta = \Psi_{YY}^{(3)} - 2B\left(u_c'''/u_c' + k^2\right)\cos\theta$, we write them in the form

$$\mathscr{L}\zeta = Ju_c'^2P_x^{(1)} - 2(u_c'''/u_c')\left(\Omega B_S + c\dot{B}_C\right), \quad \tilde{\mathscr{L}}P^{(1)} = 2kB_S, \tag{2.8}$$

where

$$\mathscr{L} = c\frac{\partial}{\partial\xi} + \left(u_c' Y - \Omega/k\right)\frac{\partial}{\partial x} + 2kB\sin\theta\frac{\partial}{\partial Y} - \eta\frac{\partial^2}{\partial Y^2} ,$$

$$\tilde{\mathscr{L}} = c\frac{\partial}{\partial\xi} + \left(u_c' Y - \Omega/k\right)\frac{\partial}{\partial x} + 2kB\sin\theta\frac{\partial}{\partial Y} - \frac{\eta}{Pr}\frac{\partial^2}{\partial Y^2} = \mathscr{L} + \frac{Pr-1}{Pr}\eta\frac{\partial^2}{\partial Y^2} .$$

Equations (2.8) (cf. (2.12) and (2.8) in C&S95) are the basic equations defining the solution inside the CL and its contribution to MSC (2.6). It is convenient to omit the supercript of $P^{(1)}$ and represent the vorticity $\zeta$ as the sum of the stratified ($\zeta_s$) and the unstratified ($\zeta_h$) parts: $\zeta = \zeta_s + \zeta_h$. We obtain

$$\tilde{\mathscr{L}} P = 2kB\sin\theta , \tag{2.9a}$$

$$\mathscr{L}\zeta_s = J u_c'^2 P_x , \tag{2.9b}$$

$$\mathscr{L}\zeta_h = -2\frac{u_c'''}{u_c'}\left[\left(\Omega - c\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right)B\sin\theta + c\frac{\mathrm{d}B}{\mathrm{d}\xi}\cos\theta\right] , \tag{2.9c}$$

MSC (2.6) is conveniently written as

$$\frac{B}{k}\left(I_0\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} - \Omega I_1\right) = \mathop{⨍}_{-\infty}^{\infty}\mathrm{d}Y\,\langle\zeta\cos\theta\rangle \equiv \mathop{⨍}_{-\infty}^{\infty}\mathrm{d}Y\,\langle\,(\zeta_s+\zeta_h)\cos\theta\rangle , \tag{2.10a}$$

$$\frac{I_0}{k}\frac{\mathrm{d}B}{\mathrm{d}\xi} = \mathop{⨍}_{-\infty}^{\infty}\mathrm{d}Y\,\langle\zeta\sin\theta\rangle \equiv \mathop{⨍}_{-\infty}^{\infty}\mathrm{d}Y\,\langle\,(\zeta_s+\zeta_h)\sin\theta\rangle , \tag{2.10b}$$

where $\mathop{⨍}_{-\infty}^{\infty}\mathrm{d}Y(\cdots) = \lim_{Z\to\infty}\int_{-Z}^{Z}\mathrm{d}Y(\cdots)$, $\quad \langle\cdots\rangle = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta(\cdots)$, $\quad I_0 = cI_1 - 2k^2 I_2$ .

The comparison of equations (2.9) and (2.10) with equations (2.13), (2.15) from C&S95 shows that the problems considered in both cases are closely related. Noteworthy are two main differences of equations (2.9*a–c*) from similar equations (2.13*a–c*) from C&S95.

In the first place, in a stratified medium two dissipation mechanisms operate on equal terms: diffusion of momentum (viscosity) and diffusion of density, and in the general case ($Pr \neq 1$) diffusion rates of the regular ($\zeta$) and the singular ($P$) disturbance components are different, unlike C&S95 where only the viscosity is at work and the diffusion rates are the same. From (2.9*a, b*) we obtain

$$\mathscr{L}G = -\frac{Pr-1}{Pr}\eta J u_c' P_{YYY} , \quad \text{where} \quad G = \zeta_s + J u_c' P_Y . \tag{2.11}$$

Since, as is easy to see, $\int\mathrm{d}Y\,\langle\zeta_s(\sin\theta,\cos\theta)\rangle = \int\mathrm{d}Y\,\langle G(\sin\theta,\cos\theta)\rangle$, equation (2.11) makes degeneracy evident, which manifested itself also in earlier research (for example, Churilov & Shukhman 1987*b*; 1988): with the diffusion rates of momentum and density being the same ($Pr = 1$), the main nonlinear contribution to the evolution equations (the right-hand sides of MSC (2.10*a,b*)), caused by stratification, vanishes.

Secondly, the right-hand side of (2.9*b*) involves the derivative with respect to $x$, but not with respect to $\xi$, as in C&S95. Because of this, in the nonlinear CL regime the interaction of the regular and the singular components of a disturbance is not driven out into outer diffusive layers, as in C&S95, and the result of this interaction has a different parity (see §4 and the Appendix).

## 3. Evolution in the viscous and unsteady CL regimes

With scaling (2.3), values of $B \ll 1$ correspond to the viscous and unsteady CL regimes. The solution of the inner problem (2.9*a,c*), (2.11) in these regimes is constructed in the form of an expansion in powers of $B$, and it is convenient to pass to the complex amplitude $\hat{A} = B\exp(\mathrm{i}\Theta)$.

It is well known (e.g. Churilov & Shukhman 1987*a*; C&S95) that in the case of two-dimensional disturbances the 'unstratified' nonlinearity in the viscous and unsteady CL regimes is non-competitive except for a narrow region of small supercriticalities ($\gamma_L < \nu$). Therefore, it will suffice here only to have a linear (in $B$) contribution of $\zeta_h$ to the right-hand sides of MSC (2.10*a,b*) which is equivalent to an appropriate indentation (from below when $u_c' > 0$) of the point $y = y_c$ in the integral $I_1$.

Thus, it will suffice to restrict ourselves to solving equations (2.9*a*), (2.11). As a result of conventional calculations (e.g. Wu *et al.* 1993; Churilov & Shukhman 1994) we obtain

$$\frac{\mathrm{i}}{k}\tilde{I}_0\frac{\mathrm{d}\hat{A}}{\mathrm{d}\xi} + \frac{\Omega}{k}\tilde{I}_1\hat{A} = 2\pi J\eta\,\frac{Pr-1}{Pr}\,\frac{k^7 u_c'^5}{c^8}\int_0^\infty \mathrm{d}\xi_1\int_0^\infty \mathrm{d}\xi_2 \mathscr{R}(\xi_1,\xi_2;Pr)\hat{A}(\xi-\xi_1)\hat{A}(\xi-\xi_1-\xi_2)\overline{\hat{A}}(\xi-2\xi_1-\xi_2). \tag{3.1}$$

Here

$$\begin{aligned}\mathscr{R}(\xi_1,\xi_2;Pr) = \xi_1^2\Bigg\{&\exp\left[-\alpha\xi_1^2\left(\frac{2}{3}\xi_1+\xi_2\right)\right]\int_0^{\xi_1}\mathrm{d}\xi_3\xi_3^3\exp\left(\alpha\frac{Pr-1}{3Pr}\xi_3^3\right)\\ &+\exp\left[-\frac{\alpha}{Pr}\xi_1^2\left(\frac{2}{3}\xi_1+\xi_2\right)\right]\int_0^{\xi_1}\mathrm{d}\xi_3\xi_3^3\exp\left(-\alpha\frac{Pr-1}{3Pr}\xi_3^3\right)\\ &+\xi_1^3\exp\left[-\alpha\xi_1^2\left(\frac{Pr+1}{3Pr}\xi_1+\xi_2\right)\right]\int_0^{\xi_2}\mathrm{d}\xi_3\exp\left(\alpha\frac{Pr-1}{Pr}\xi_1^2\xi_3\right)\Bigg\},\end{aligned} \tag{3.2}$$

where $\alpha = \eta k^2 u_c'^2/c^3$, $\tilde{I}_0 = c\tilde{I}_1 - 2k^2 I_2$ and the tilde over $I_1$ means that the integral is evaluated with the indentation of $y = y_c$ from below.

This is just the evolution equation describing the development of disturbances in the viscous and unsteady CL regimes, including the transition region between them where $l_t = O(l_\nu)$. In limiting cases ($l_t \ll l_\nu$ and $l_t \gg l_\nu$) it becomes significantly simpler.

### 3.1. *Viscous CL regime* ($l_\nu \gg l_t$)

This case corresponds to the limit $\alpha \to \infty$. Because of a rapid exponential decrease, the main contribution to the integral on the right-hand side of (3.2) is made by small delays. The main term of the asymptotic expansion does not go to zero and therefore the main nonlinearity is local in $\xi$ (unlike C&S95, equation (4.1); see also Churilov & Shukhman 1994, §4.2.1):

$$\frac{\mathrm{i}}{k}\tilde{I}_0\frac{\mathrm{d}\hat{A}}{\mathrm{d}\xi} + \frac{\Omega}{k}\tilde{I}_1\hat{A} = -4a_4(Pr)(Pr-1)J\frac{k^5 u_c'^3}{c^5\alpha^{5/3}}|\hat{A}|^2\hat{A}; \tag{3.3}$$

the non-local nonlinearity, however, is of $O(\alpha^{-2})$. Coefficient $a_4(Pr) < 0$ and its explicit form is given in C&S96.

Note that the right-hand side of (3.3) is real, and its sign is determined by the sign of $(Pr-1)$. Since (see (2.6)) $\tilde{I}_1 = I_1 + \mathrm{i}\pi u_c'''/u_c'^2$ and $u_c''' < 0$, the real part of the coefficient

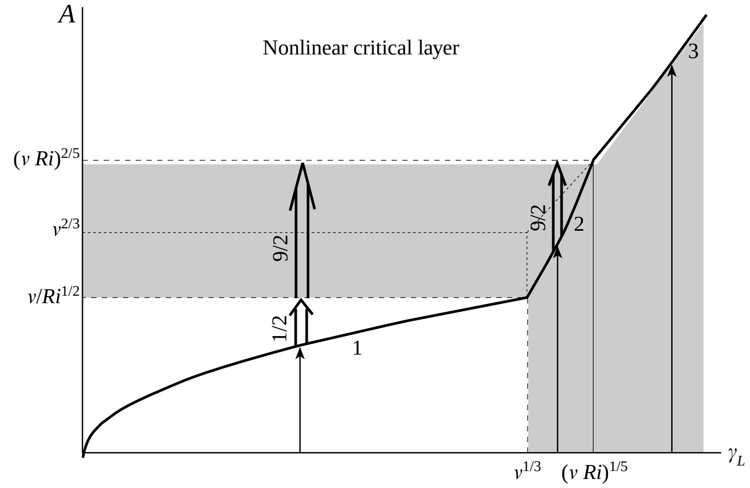


FIGURE 2. Evolution diagram in the viscous and unsteady CL regimes (before the transition into the nonlinear CL regime). Heavy lines show nonlinearity thresholds: 1, $A_1 = \left(\gamma_L \nu^{5/3}/Ri\right)^{1/2}$; 2, $A_2 = \left(\gamma_L^9/(\nu Ri)\right)^{1/2}$; 3, $A_3 = \gamma_L^2$. The arrows show the various stages of evolution: $\longrightarrow A \sim \exp(\gamma_L \xi)$; $\stackrel{\alpha}{\Longrightarrow} A \sim (\xi_0 - \xi)^{-\alpha}$. The unsteady CL region is shaded.

at $\mathrm{d}\hat{A}/\mathrm{d}\xi$ is positive. By extracting the supercriticality-induced correction to the wave number, $\hat{A} = \tilde{A}\exp(\mathrm{i}K\xi)$, $K = \mathrm{Re}(\tilde{I}_1/\tilde{I}_0)\Omega$, and returning to the 'physical' variables, $A = \varepsilon\tilde{A}$, $\nu = \varepsilon^{3/2}\eta$, $Ri = \varepsilon J$, $x = \varepsilon^{1/2}\xi$, we write the evolution equation (3.3) in a brief form:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A + d_1 \frac{(Pr-1)Ri}{\nu^{5/3}}|A|^2 A, \tag{3.4}$$

where $\gamma_L = -\varepsilon^{1/2}\mathrm{Im}(\tilde{I}_1/\tilde{I}_0)\Omega$, $d_1 = O(1)$, $\mathrm{Re}(d_1) > 0$. Thus, as would be expected (see Churilov & Shukhman 1987), the nonlinearity has a stabilizing effect when $Pr < 1$ and a destabilizing effect when $Pr > 1$. The nonlinearity becomes competitive when

$$A \sim A_1 = O\left(\gamma_L^{1/2} Ri^{-1/2} \nu^{5/6}\right) \tag{3.5}$$

(curve 1 in figure 2). The threshold of nonlinearity (3.5) lies below the formal boundary of the nonlinear CL ($A \sim \nu^{2/3}$) throughout the region of the viscous CL ($\gamma_L < \nu^{1/3}$) if stratification is strong enough,

$$Ri > \nu^{2/3}. \tag{3.6}$$

The case of weaker stratification was studied in our earlier publication C&S96; therefore, the inequality (3.6) will be taken to be satisfied in what follows.

The evolution behaviour after the attainment of the threshold of nonlinearity (3.5) depends on the Prandtl number: when $Pr < 1$ the disturbance is stabilized on this level, and when $Pr > 1$ it continues to increase further, but now in an explosive way, however, as

$$|A| \sim \left[\nu^{-5/3} Ri\,(x_0 - x)\right]^{-1/2} \tag{3.7}$$

(the bottom broad arrow in figure 2). Now the unsteady scale $l_t$ increases together with the amplitude, $l_t \sim |A|^2 Ri/\nu^{5/3}$, and when $|A| = O(\nu/Ri^{1/2}) \ll \nu^{2/3}$ it equals the viscous scale $l_\nu \sim \nu^{1/3}$: the transition into the unsteady CL regime occurs (see the next subsection).

### 3.2. *Unsteady CL regime ($l_t \gg l_\nu$)*

This case corresponds to the limit $\alpha \to 0$ and NEE (3.2) takes the form

$$\frac{\mathrm{i}}{k}\tilde{I}_0\frac{\mathrm{d}\hat{A}}{\mathrm{d}\xi} + \frac{\Omega}{k}\tilde{I}_1\hat{A} = \pi\eta J\frac{Pr-1}{Pr}\frac{k^7 u_c'^5}{c^8}\int_0^\infty \mathrm{d}\zeta\, \zeta^7 \int_0^1 \mathrm{d}\sigma\, \sigma^5(2-\sigma)\hat{A}(\xi-\zeta)\hat{A}(\xi-\sigma\zeta)\overline{\hat{A}}(\xi-(1+\sigma)\zeta), \quad (3.8)$$

usual for evolution equations in the unsteady CL regime (e.g. Hickernell 1984; Churilov & Shukhman 1988). Let (3.8) be also represented in brief form in terms of physical variables (cf. (4.3) from C&S95)

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A + d_2 Ri\,(Pr-1)\nu\, \mathrm{e}^{-\mathrm{i}\chi}\int_0^\infty \mathrm{d}s\, s^7 \int_0^1 \mathrm{d}\sigma\, \sigma^5(2-\sigma)A(x-s)A(x-\sigma s)\overline{A}(x-(1+\sigma)s), \quad (3.9)$$

where $0 < d_2 = O(1)$, $|\chi| < \pi/2$. The threshold of nonlinearity

$$A \sim A_2 = O\left(\gamma_L^{9/2}\nu^{-1/2}Ri^{-1/2}\right) \quad (3.10)$$

(curve 2 in figure 2) intersects the formal boundary of the nonlinear CL ($|A| \sim \gamma_L^2$) when

$$\gamma_L \sim (\nu Ri)^{1/5}. \quad (3.11)$$

With a greater supercriticality, the nonlinearity in the unsteady CL regime is non-competitive, and exponential growth of unstable disturbances terminates when $A \sim O\left(\gamma_L^2\right)$ with the transition into the nonlinear CL regime. If, however, $\nu^{1/3} < \gamma_L < (\nu Ri)^{1/5}$, then after reaching the threshold of nonlinearity (3.10), exponential growth is replaced by an explosive one (the right-hand broad arrow in figure 2), *no matter what the sign of* $(Pr-1)$. The asymptotic law of growth

$$A \sim (\nu Ri)^{-1/2}(x_1 - x)^{-9/2+\mathrm{i}\beta}, \quad \beta = \beta(\varphi), \quad \varphi = \chi + (1-\mathrm{sign}(\mathrm{Pr}-1))\pi/2 \quad (3.12)$$

assumes (when $\varphi \neq 0$) an explosive growth not only of the amplitude but also the phase $\Theta$.

Because of the ambiguity of the function $\beta(\varphi)$ (e.g. Shukhman 1991), it is unclear *a priori* precisely what branch will be reached by the solution of (3.9) when $x \to x_1$; the value of $x_1$ is also determined numerically only. For a numerical solution, equation (3.9) is conveniently brought to a 'universal' form. When $x \to -\infty$ the disturbance grows exponentially, $A = A_0\exp(\gamma_L x)$. We put $A(x) = A_0 a(x)\exp(\gamma_L x)$, $a(x) \to 1$ when $x \to -\infty$, and introduce a new evolution variable $T = T_0\exp(2\gamma_L x)$, $T_0 = d_2|Pr-1|\nu Ri(2\gamma_L)^{-9}|A_0|^2$. From (3.9) we obtain the equation

$$\frac{\mathrm{d}a}{\mathrm{d}T} = \mathrm{e}^{-\mathrm{i}\varphi}\int_0^\infty \mathrm{d}t\, t^7\mathrm{e}^{-t}\int_0^1 \mathrm{d}\sigma\, \frac{\sigma^5(2-\sigma)}{(1+\sigma)^8}\, a\left(T\mathrm{e}^{-t/(1+\sigma)}\right) a\left(T\mathrm{e}^{-\sigma t/(1+\sigma)}\right) \overline{a}\left(T\mathrm{e}^{-t}\right), \quad a(0) = 1, \quad (3.13)$$

in which only the phase $\varphi$ is dependent on flow structure. For the flow $u = 1 + \tanh y$ we have $\chi = \arctan(\pi/2) - \pi/2 \approx -0.1805\pi$, i.e. $\varphi = -0.1805\pi$ for $Pr > 1$ and $\varphi = 0.8195\pi$ for $Pr < 1$.

The results obtained by solving equation (3.13) are presented in figures 3 and 4. As in C&S95, an explosive growth of (3.12) is not a final stage of the weakly nonlinear evolution of disturbances: because of the weakness of the stratification the unsteady

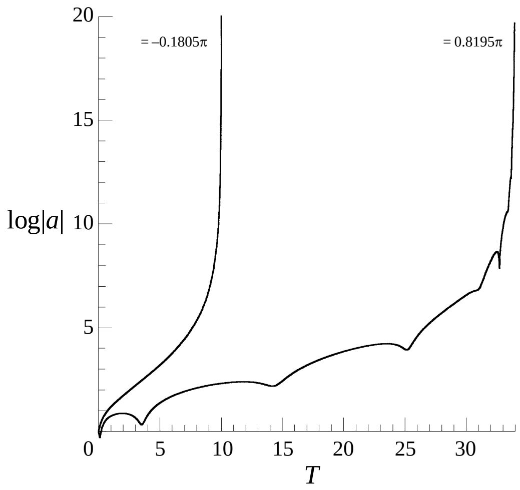


FIGURE 3. Wave amplitude vs. logarithmic time $T$ in the unsteady CL regime for two values of the parameter $\varphi$: $\varphi = \arctan(\pi/2) \mp \pi/2$ corresponding to $\mathrm{sign}(Pr-1) = \pm 1$ in the model $u = 1 + \tanh y$.

scale $l_t \sim \left(|A|^2 \nu Ri\right)^{1/9}$ increases insufficiently fast with increasing amplitude and when

$$A \sim (\nu Ri)^{2/5} \tag{3.14}$$

the nonlinear scale $l_N \sim |A|^{1/2}$ overtakes it: the transition into the nonlinear CL regime occurs (see figure 2). Note that when $\gamma_L < (\nu Ri)^{1/5}$ this transition proceeds from the explosive growth stage in the unsteady CL regime†.

To conclude this Section, one remark is in order. We have analysed only limiting cases of NEE (3.1). Therefore, near boundaries between viscous and unsteady CL regions in amplitude–growth rate space the results obtained are valid qualitatively only. For a quantitative description of the evolution, it is necessary to solve the full equation (3.1) as has been done for some other problems (e.g. Goldstein & Leib 1989; Leib 1991; Wu & Cowley 1995). Not only do such solutions serve to confirm results of asymptotic analysis, but they make it possible to determine more precisely the boundary in the supercriticality between the viscous and the unsteady types of evolution (which is of particular interest at $Pr < 1$ when the saturation of an instability at the level (3.5) is next to a relatively complicated development with an explosive asymptotic growth, see figures 3 and 4).

Nevertheless, we shall restrict our consideration to an asymptotic analysis and concentrate on a more interesting (and essential in this problem) numerical investigation of the transition to the nonlinear CL regime (§ 5).

## 4. Quasi-steady evolution in the nonlinear CL regime

A quasi-steady state in the nonlinear CL regime means smallness of the evolution term in the operators $\mathscr{L}$ and $\tilde{\mathscr{L}}$ compared not only with the nonlinear term but also with the viscous term, which corresponds to the inequalities

$$l_t \ll l_\nu^3/l_N^2 \ll l_N \quad \text{or} \quad \Gamma \ll \eta/B \ll B^{1/2}; \quad \Gamma = \left|B^{-1}\mathrm{d}B/\mathrm{d}\xi\right| . \tag{4.1}$$

† It will be recalled that if $\gamma_L < \nu^{1/3}$ the unsteady CL regime is realized only when $Pr > 1$; when $Pr < 1$ the amplitude ceases to increase in the viscous CL regime at the level (3.5).

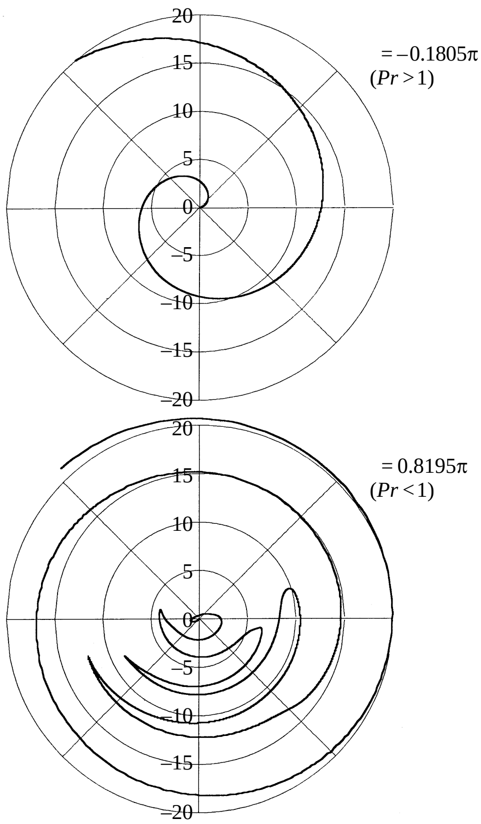


FIGURE 4. Evolution path of the disturbance: $\log|a(T)|$ is plotted in the radial direction, the polar angle is equal to $\arg[a(T)]$ .

On the other hand, in outer ($|y - y_c| = O(1)$) flow regions the dissipation terms are small compared with the unsteady ones ($\gamma_L \gg v$). Therefore, the solution inside the CL has to be matched to (2.5) not directly but through solutions in so-called outer diffusive layers (ODL), (e.g. C&S95) having a scale

$$l_{ODL} \sim (\varepsilon\eta/\Gamma)^{1/2} \sim (l_v^3/l_t)^{1/2} ,$$

where dissipation and unsteady terms are of the same order. The left-hand side of each of the inequalities (4.1) does mean that ODLs (together with unsteady processes occurring in them) lie at the far periphery of the CL: $l_{ODL} \gg l_N$.

Equations (2.9*a–c*) are conveniently written in terms of the variables $\theta$, $\tau = \xi/c$ and $z = (u_c'/2B)^{1/2}(Y - Y_c(\tau))$, where $Y_c(\tau) = (\Omega - \Theta_\tau)/(ku_c')$ is the displacement of the critical level of the neutral mode and subscript $\tau$ denotes the derivative in $\tau$. We obtain

$$\mathscr{M}\left(\frac{\lambda}{Pr}\right)P = -\frac{1}{k(2Bu_c')^{1/2}}\mathscr{T}P + \left(\frac{2B}{u_c'}\right)^{1/2}\sin\theta, \tag{4.2a}$$

$$\mathscr{M}(\lambda)\zeta_s = -\frac{1}{k(2Bu_c')^{1/2}}\mathscr{T}\zeta_s + J\left(\frac{u_c'^{\,3}}{2B}\right)^{1/2}P_\theta, \tag{4.2b}$$

$$\mathscr{M}(\lambda)\zeta_h = -\frac{1}{k(2Bu_c')^{1/2}}\mathscr{T}\zeta_h - \frac{2u_c'''}{k(2Bu_c'^3)^{1/2}}\left[(\Omega-\Theta_\tau)B\sin\theta + B_\tau\cos\theta\right], \quad (4.2c)$$

where

$$\mathscr{M}(\mu) = z\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial z} - \mu\frac{\partial^2}{\partial z^2}, \quad \mathscr{T} = \frac{\partial}{\partial\tau} + \frac{\Theta_{\tau\tau}}{k(2Bu_c')^{1/2}}\frac{\partial}{\partial z} - \frac{B_\tau}{2B}z\frac{\partial}{\partial z}, \quad \lambda = \frac{\eta(u_c')^{1/2}}{k(2B)^{3/2}}\ .$$

In the first approximation unsteady terms (terms with $\mathscr{T}$ on the right-hand sides of (4.2*a–c*)) may be neglected. It is convenient to introduce the functions (e.g. C&S96) $g_1(\lambda;z,\theta)$, $g_2(\lambda;z,\theta)$, $g_4(\lambda,Pr;z,\theta)$, satisfying the equations

$$\mathscr{M}(\lambda)g_1 = -2\sin\theta\ , \quad \mathscr{M}(\lambda)g_2 = -2\cos\theta\ , \quad \mathscr{M}(\lambda)g_4 = -\partial g_1\left(\lambda/Pr;z,\theta\right)/\partial\theta\ ; \quad (4.3)$$

$$\partial g_i/\partial z \to 0 \ \text{ as } \ z\to\pm\infty, \quad i=1,2,4,$$

and represent the solution of (4.2*a–c*) in terms of them:

$$\left.\begin{aligned} P = -\left(B/2u_c'\right)^{1/2}g_1\left(\lambda/Pr;z,\theta\right)+\cdots\ , \quad \zeta_s = Ju_c'\,g_4(\lambda,Pr;z,\theta)/2+\cdots, \\ \zeta_h = \frac{u_c'''}{k(2Bu_c'^3)^{1/2}}\left[(\Omega-\Theta_\tau)Bg_1(\lambda;z,\theta)+B_\tau g_2(\lambda;z,\theta)\right]+\cdots. \end{aligned}\right\} \quad (4.4)$$

Dots denote contributions made by unsteady terms. In the limit $\lambda \ll 1$ dictated by the inequalities (4.1), the spatial structure of these functions is written in Appendix A of C&S96. In the limiting case of vanishing viscosity the functions $g_1$ and $g_2$ are continuous on the cat's eye boundaries $\kappa \equiv z^2/2+\cos\theta = 1$, while $g_4$ undergoes a discontinuity here:

$$g_4(1+0)-g_4(1-0) \equiv [g_4] = \pi\left\{|\sin(\theta/2)| - (\pi/4)(1-1/Pr)\right\}\ . \quad (4.5)$$

Continuity of $g_1$ means continuity of the density $P$ on the cat's eye boundary, and the jump of $g_4$ implies a jump of vorticity $\zeta$†

$$[\zeta] = (Ju_c'/2)[g_4]\ .$$

This jump is caused entirely by stratification because (as was shown by Haberman as early as 1972) in a homogeneous flow the vorticity on the cat's eye boundary is continuous. It is natural to call it the baroclinic jump of vorticity. Note that in the nonlinear quasi-steady CL regime the jump (in the first approximation) is independent of the wave's amplitude and is determined entirely by the Richardson number.

Contributions to the right-hand sides of MSCs (2.10*a,b*) are expressed in terms of the functions

$$\Phi_1(\lambda) = \fint_{-\infty}^{\infty}\mathrm{d}z\langle g_1\sin\theta\rangle, \quad \Phi_2(\lambda) = \fint_{-\infty}^{\infty}\mathrm{d}z\langle g_2\cos\theta\rangle, \quad \Phi_4(\lambda,Pr) = \fint_{-\infty}^{\infty}\mathrm{d}z\langle g_4\cos\theta\rangle$$

as

$$\left.\begin{aligned} \fint_{-\infty}^{\infty}\mathrm{d}Y\,\langle\zeta_h\sin\theta\rangle = \frac{u_c'''}{ku_c'^2}\,\Phi_1(\lambda)\left(\Omega - c\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right)B, \quad \fint_{-\infty}^{\infty}\mathrm{d}Y\,\langle\zeta_h\cos\theta\rangle = \frac{u_c'''}{ku_c'^2}\,\Phi_2(\lambda)c\frac{\mathrm{d}B}{\mathrm{d}\xi}, \\ \fint_{-\infty}^{\infty}\mathrm{d}Y\,\langle\zeta_s\cos\theta\rangle = J\left(Bu_c'/2\right)^{1/2}\Phi_4(\lambda,Pr); \quad \fint_{-\infty}^{\infty}\mathrm{d}Y\,\langle\zeta_s\sin\theta\rangle = 0. \end{aligned}\right\} \quad (4.6)$$

Recall that the $\cos\theta$ contribution from $g_1$ and $\sin\theta$ from $g_2$ and $g_4$ vanish due to parity of these function (for details see C&S96).

† Of course, the full solution for any $\lambda$ is continuous and continuous transition takes place in a narrow transition layer with width $\lambda^{1/2}$ (see C&S96, Appendix A).

Note that, according to equation (4.2*b*), the interaction of the regular and the singular components of a disturbance occurs throughout the entire CL thickness and, as is apparent from (4.6), makes a local (dependent on the value of the amplitude $B$ only at a given point $\xi$) $\cos\theta$ contribution to the MSC, unlike C&S95 where such an interaction was ousted into external diffusive layers and made a non-local (determined by the entire past evolution) $\sin\theta$ contribution. It is this that determines the cardinal differences of the evolution character in the nonlinear CL regime in these two so closely related problems to be shown later in the text.

On substituting (4.6) into the right-hand sides of (2.10*a,b*) and separating $\mathrm{d}B/\mathrm{d}\xi$ and $\mathrm{d}\Theta/\mathrm{d}\xi$, we obtain a system of nonlinear evolution equations

$$\left.\begin{aligned} \frac{\mathrm{d}B}{\mathrm{d}\xi} &= -\frac{2k^2u_c'''}{u_c'^2}\left[I_2\Omega B + \frac{cJ}{4k}(2Bu_c')^{1/2}\Phi_4(\lambda,Pr)\right]\frac{\Phi_1(\lambda)}{\Delta(\lambda)}, \\ \frac{\mathrm{d}\Theta}{\mathrm{d}\xi} &= \frac{\Omega}{c}\left\{1 + \frac{2k^2I_0}{\Delta(\lambda)}\left[I_2 + \frac{cJ}{2k\Omega}\left(\frac{u_c'}{2B}\right)^{1/2}\Phi_4(\lambda,Pr)\right]\right\}, \end{aligned}\right\} \tag{4.7}$$

$$\Delta(\lambda) \equiv I_0^2 + c^2\left(u_c'''/u_c'^2\right)^2\Phi_1(\lambda)\Phi_2(\lambda)\ .$$

Using the asymptotic expansions of $\Phi_i$ in the limit $\lambda \ll 1$ (e.g. C&S96) we obtain the desired evolution equations in the quasi-steady nonlinear CL regime:

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = -\eta\frac{C^{(1)}}{\Delta(0)}\frac{u_c'''}{u_c'}\left(\frac{kI_2\Omega}{(2Bu_c')^{1/2}} + \frac{Pr-1}{4Pr}\frac{cC^{(4)}}{B}J\right), \tag{4.8a}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = \left(1 + \frac{2k^2I_0I_2}{\Delta(0)}\right)\frac{\Omega}{c} + \frac{Pr-1}{Pr}\left(\frac{u_c'}{2B}\right)^{1/2}\frac{kI_0C^{(4)}}{\Delta(0)}J, \tag{4.8b}$$

where $\Delta(0) = I_0^2 + c^2\left(u_c'''/u_c'^2\right)^2C^{(1)}C^{(2)}$, $C^{(1)} = -5.5151\ldots$, $C^{(2)} = -2.5008\ldots$, $C^{(4)} = -1.9377\ldots$.

In the subsequent analysis it is convenient to pass to 'physical' variables, and NEE (4.8*a*) is conveniently written in brief form

$$\frac{\mathrm{d}A}{\mathrm{d}x} = v\left(d_3\frac{\gamma_L}{A^{1/2}} + d_4\frac{Pr-1}{Pr}\frac{Ri}{A}\right), \quad d_3 > 0, \quad d_4 > 0\ . \tag{4.9}$$

The first term on the right-hand side of (4.9) appears in all problems concerning the evolution of unstable disturbances in the nonlinear CL regime and is the result of a reduction of the linear growth rate $\gamma_L$, and the second term is caused by the stratification influence (in the viscous CL regime this term turns into the nonlinear term of NEE (3.4), see also C&S96).

We now determine the validity range of the evolution equations obtained. Quasi-steady-state conditions (4.1) are satisfied if

$$\max(\gamma_L/A^{1/2}, Ri/A) \ll 1 \quad \text{or} \quad A \gg \max(Ri, \gamma_L^2)\ ,$$

i.e. in the region which is dark shaded in figure 5. It is easy to see that for a not too small stratification ($Ri \gg v^{2/3}$)), the gap

$$(vRi)^{2/5} < A < Ri, \quad \gamma_L^2 < A \tag{4.10}$$

(lightly shaded in figure 5) lies between the level at which the transition to the nonlinear CL regime from the unsteady CL regime occurs and the validity range of

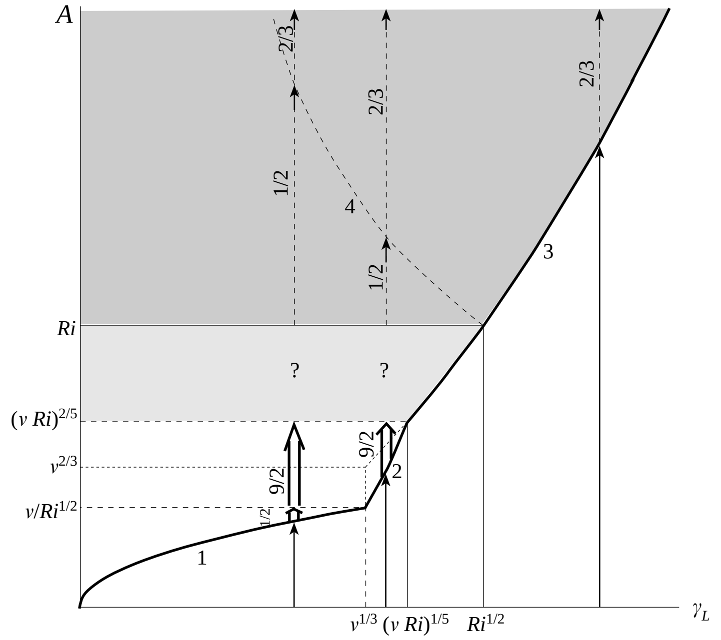


FIGURE 5. Diagram of disturbance development in all possible CL regimes. Designations are the same as in figure 2; $\overset{\beta}{- - \rightarrow}$ – $A \sim \xi^{\beta}$. Curve 4 – $A_4 = \left(Ri/\gamma_L\right)^2$ – bounds from below the region where stratification is unimportant. Dark shading shows the region of a quasi-steady nonlinear CL; the gap of an essentially unsteady evolution in the nonlinear CL regime is lightly shaded. The absence of a simple analytic expression describing the disturbance dynamics in this gap is marked by ?.

NEE (4.8). In this gap the development of disturbances in the nonlinear CL regime is essentially unsteady. To understand what happens to the disturbance – will it transit through this gap and reach the regime of quasi-steady evolution or not – one has to have a simultaneous numerical solution of equations (2.9*a–c*) and (2.10*a, b*). This issue is taken up in the next Section. It should be noted that in problems treated so far no such situation has been encountered: unsteady processes underwent a reasonably fast relaxation (e.g. G&H), and the disturbance could be thought of as passing from the unsteady CL regime directly into the *quasi-steady* nonlinear CL regime (see the Introduction in C&S95).

Before embarking on a numerical investigation of the evolution we shall try to predict it using equations (4.9) and based on the fact that in order for the gap (4.10) to be 'surmounted' it is necessary to have a growth of amplitude. Terms on the right-hand side of (4.9) have the same order of magnitude when $A = O\left((Ri/\gamma_L)^2\right)$, i.e. on curve 4 (see figure 5) which runs above the gap (4.10). Consider the part of the dark shaded region which is adjacent to the gap. Here, in NEE (4.9) the main term is the second ('stratified') term. Depending on the Prandlt number, two fundamentally different situations are possible.

(i) When $Pr < 1$ the stratification contribution is negative, which corresponds to a decrease of amplitude $A$. Consequently, when $Pr < 1$ a quasi-steady nonlinear CL is evolutionarily unattainable: even if the disturbance manages to come closer to it 'from below', it will at once be forced out backward. Hence a growth of the amplitude is bounded by the level $A \sim Ri$, and one can speak of the stabilization of instability by nonlinear processes when $Pr < 1$ not only in the viscous CL regime ($\gamma_L < \nu^{1/3}$) but also in a more extensive range of supercriticalities $\gamma_L < Ri^{1/2}$.

(ii) When $Pr > 1$ the right-hand side of (4.9) is positive, and the disturbance that has come 'from below' increases further according to a power law

$$A \sim (\nu Ri\, x)^{1/2} \tag{4.11}$$

up to the curve 4, $A \sim (Ri/\gamma_L)^2$, where the first ('unstratified') term on the right-hand side of (4.9) becomes the main term, and the evolution law changes to a 'classical' one

$$A \sim (\nu\gamma_L\, x)^{2/3} \ . \tag{4.12}$$

This law occurs everywhere in a weakly nonlinear region ($A < 1$) above curves 3 and 4 in figure 5.

Note that in the case of a weaker stratification, $Ri < \nu^{2/3}$, the gap (4.10) disappears, and the transition occurs virtually at once into the quasi-steady nonlinear CL regime; this case was investigated earlier in paper C&S96.

As has already been pointed out earlier in this Section, the CL quasi-steady condition means that unsteady processes are forced out to the CL periphery. The evolution equations (4.8) do not take these processes into account. However, the unsteadiness that was forced out to the ODL gives corrections to these equations, and hence also to the evolution law described by them, of $O\big((\Gamma B/\eta)^{1/2}\big) \sim O(l_N/l_{ODL})$, and these corrections can be numerically not small and important, for example, when comparing results of numerical calcualtions with the asymptotic solution. As done in G&H, we have taken into consideration the unsteadiness in the ODL and obtained more precise evolution equations. Their derivation and explicit form are given in the Appendix.

## 5. Numerical results

### 5.1. *Transformation of equations and numerical procedure*

A numerical solution of the system of equations (2.8), (2.10) was performed for a flow with the velocity profile $u = c + \tanh y$ which was also considered in G&H. For it $\omega_{cr} = c$, $k = u_c' = 1$, $u_c''' = -2$, $I_1 = 0$, $I_2 = 2$. In order to be able to make a direct comparison of our calculations with those in G&H, we now pass to analogous variables. We introduce

$$\left.\begin{aligned} \bar{\xi} = \tfrac{1}{2}|\Omega|\xi, \quad \tilde{Y} = \frac{Y-\Omega}{c\,(|\Omega|/2)}, \quad C(\bar{\xi}) = \frac{2B}{(c\Omega/2)^2}\mathrm{e}^{\mathrm{i}\Theta(\bar{\xi})}, \\ \tilde{\zeta} = \frac{\zeta}{c\Omega^2}, \quad \tilde{P} = \frac{P}{c|\Omega|/2}, \quad \bar{\lambda} = \frac{\eta}{(c|\Omega|/2)^3}, \quad g = \frac{Jc}{4(\eta/Pr)^{2/3}}, \end{aligned}\right\} \tag{5.1}$$

and choose the origin in $X(\equiv x - (c + \varepsilon^{1/2}\Omega)t)$ and $\bar{\xi}$ such that when $\bar{\xi} \to -\infty$ the disturbance is of the form

$$\tilde{P} = \mathrm{Re}\left\{\Pi(\tilde{Y})\mathrm{e}^{q\bar{\xi}+\mathrm{i}X}\right\}, \quad \tilde{\zeta} = \mathrm{Re}\left\{f(\tilde{Y})\mathrm{e}^{q\bar{\xi}+\mathrm{i}X}\right\}. \tag{5.2}$$

Here $q = \pi/(1 + \mathrm{i}\pi c/2)$, and $\Pi(\tilde{Y})$ and $f(\tilde{Y})$ are the solutions of a linearized inner problem:

$$\Pi(\tilde{Y}) = -\bar{\lambda}^{-1/3}\Phi\left(\bar{\lambda}^{-1/3}(\tilde{Y} - \mathrm{i}q)\right) \ , \tag{5.3}$$

$$f(\tilde{Y}) = (q/\pi)\bar{\lambda}^{-1/3}\Phi\left(\bar{\lambda}^{-1/3}(\tilde{Y} - \mathrm{i}q)\right) + gG\left[\left(\bar{\lambda}/Pr\right)^{-1/3}(\tilde{Y} - \mathrm{i}q); Pr\right] \ , \tag{5.4}$$

where

$$\Phi(z) = \mathrm{i}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t^3/3-\mathrm{i}tz}, \quad G(z\,;\eta) = \int_0^\infty \mathrm{d}t\, t\, \mathrm{e}^{-\mathrm{i}tz}\int_0^1 \mathrm{d}v\, v^3 \exp\left\{-\tfrac{1}{3}t^3\left[\eta-(\eta-1)v^3\right]\right\}. \quad (5.5)$$

Omitting the tilde over $P$, $\zeta$ and $Y$ and passing to Fourier harmonics

$$(\zeta, P) = \frac{1}{2}\sum_{n=-\infty}^{\infty}(\zeta_n, P_n)\mathrm{e}^{\mathrm{i}nX}, \quad (\zeta_{-n}, P_{-n}) = (\overline{\zeta}_n, \overline{P}_n)$$

we obtain a system of equations for $\zeta_n(\bar{\xi}, Y)$, $P_n(\bar{\xi}, Y)$ and the amplitude $C(\bar{\xi})$:

$$\frac{\partial\zeta_n}{\partial\bar{\xi}} + \mathrm{i}nY\zeta_n + \frac{\mathrm{i}}{2}\left(\overline{C}\frac{\partial\zeta_{n+1}}{\partial Y} - C\frac{\partial\zeta_{n-1}}{\partial Y}\right) - \bar{\lambda}\frac{\partial^2\zeta_n}{\partial Y^2} = \delta_{n1}\left(\frac{c}{2}\frac{\mathrm{d}C}{\mathrm{d}\bar{\xi}} + \mathrm{i}C\right) + \mathrm{i}ng\left(\frac{\bar{\lambda}}{Pr}\right)^{2/3} P_n, \quad (5.6)$$

$$\frac{\partial P_n}{\partial\bar{\xi}} + \mathrm{i}nYP_n + \frac{\mathrm{i}}{2}\left(\overline{C}\frac{\partial P_{n+1}}{\partial Y} - C\frac{\partial P_{n-1}}{\partial Y}\right) - \frac{\bar{\lambda}}{Pr}\frac{\partial^2 P_n}{\partial Y^2} = -\mathrm{i}C\delta_{n1}\,, \quad (5.7)$$

$$\mathrm{i}\frac{\mathrm{d}C}{\mathrm{d}\bar{\xi}} = \int\!\!\!\!\!-_{-\infty}^{\infty}\zeta_1\,\mathrm{d}Y \quad (5.8)$$

with initial conditions at $\bar{\xi}\to-\infty$

$$P_n = \delta_{n1}\exp(q\bar{\xi})\Pi(Y), \quad \zeta_n = \delta_{n1}\exp(q\bar{\xi})f(Y), \quad C = \exp(q\bar{\xi})\,. \quad (5.9)$$

Boundary conditions are imposed at $Y = \pm Y_m$, $Y_m \gg 1$, by assigning corresponding asymptotic values for $P_n$ and $\zeta_n$:

$$P_0 = -\frac{|C|^2}{Y^3} + O\left(Y^{-4}\right), \; P_1 = -\frac{C}{Y} + \frac{\mathrm{i}}{Y^2}\frac{\mathrm{d}C}{\mathrm{d}\bar{\xi}} + O\left(Y^{-3}\right), \; P_n = O\left(Y^{-(2n-1)}\right) \text{ when } n \geqslant 2; \quad (5.10)$$

$$\left.\begin{aligned} \zeta_1 &= -\frac{\mathrm{i}}{Y}\left(\frac{c}{2}\frac{\mathrm{d}C}{\mathrm{d}\bar{\xi}} + \mathrm{i}C - \frac{\mathrm{i}Cj}{Y}\right) + \frac{1}{Y^2}\left(\frac{c}{2}\frac{\mathrm{d}^2C}{\mathrm{d}\bar{\xi}^2} + \mathrm{i}\frac{\mathrm{d}C}{\mathrm{d}\bar{\xi}} - \frac{2\mathrm{i}Cj}{Y}\right) + O\left(Y^{-3} + jY^{-4}\right), \\ \zeta_0 &= \frac{c}{4}\frac{|C|^2}{Y^2} + O\left(Y^{-3} + jY^{-4}\right), \quad \zeta_n = O\left(Y^{-(2n-1)}\right)\left[1 + O\left(jY^{-1}\right)\right] \text{ when } n \geqslant 2, \end{aligned}\right\} \quad (5.11)$$

where $j \equiv g(\bar{\lambda}/Pr)^{2/3}$. We call attention to the fact that the expansion developed above assumes $j \lesssim Y$; therefore $Y_m$ must be chosen so that $jY_m^{-1}$ is of order unity or less. This same remark applies to the equation for the amplitude (5.8) transformed in view of a finite $Y_m$:

$$\begin{aligned} &\frac{\mathrm{d}C}{\mathrm{d}\bar{\xi}}\left\{1 + 2c^2 - \frac{4}{Y_m} + \frac{4}{Y_m^2} - \frac{2j}{cY_m^2}\left[3c^2 + 1 - \frac{4}{Y_m^2} - \frac{4(1+c^2)}{Y_m}\right]\right\} \\ &\quad = 2\mathrm{i}C\left[-c + j\left(\frac{1+2c^2}{Y_m} - \frac{2}{Y_m^2}\right) - \frac{2j^2}{cY_m^2}\left(1 + 2c^2 - \frac{2}{Y_m}\right)\right] \\ &\quad -\mathrm{i}I_{10}\left[1 - \frac{2}{Y_m} + c^2 - \frac{2j}{cY_m^2}\left(1 + 2c^2 - \frac{2}{Y_m}\right)\right] \\ &\quad +\mathrm{i}\frac{c}{Y_m}\left(I_{11} - jJ_{10}\right) - \mathrm{i}\left(\frac{c}{Y_m}\right)^2\left(I_{12} - \tfrac{1}{2}\overline{C}I_{20} - 2jJ_{11}\right) + O\left(Y_m^{-3} + jY_m^{-4}\right). \quad (5.12) \end{aligned}$$

Here $I_{nl} = \int_{-Y_m}^{Y_m}\zeta_n Y^l\,\mathrm{d}Y$, $\quad J_{nl} = \int_{-Y_m}^{Y_m} P_n Y^l\,\mathrm{d}Y$. Equation (5.12) generalizes equation (4.10) from G&H for the case of a weakly stratified flow.

For the numerical solution of equations (5.6), (5.7) and (5.12) with initial conditions (5.9) and boundary conditions (5.10), (5.11) an algorithm described in G&H with the following modifications was applied: (i) a grid non-uniform in $Y$† was used; (ii) as the evolution variable $\bar{\xi}$ increased, a choice of the step in $\bar{\xi}$ was continuously carried out automatically, so that a preset accuracy was achieved. The algorithm adopted was tested, in particular by comparing results of our calculations for $g = 0$ (an unstratified flow) with results reported in G&H for $c = 1$ and for different values of $\bar{\lambda}$.

### 5.2. *Results of calculations for $Pr > 1$*

The analysis in §4 suggests that when $Pr > 1$, as the amplitude increases, the development of a disturbance, sooner or later, reaches the quasi-steady nonlinear CL regime. The evolution equations, describing this stage, in the variables (5.1) have the form

$$\frac{\mathrm{d}|C|}{\mathrm{d}(\bar{\lambda}\bar{\xi})} = \tfrac{2}{3}a_\infty^{3/2}|C|^{-1/2}\left(1 + Q|C|^{-1/2}\right), \quad \frac{\mathrm{d}\Theta}{\mathrm{d}\bar{\xi}} \equiv \Theta'_{\bar{\xi}} = \Theta'_\infty\left(1 - \frac{Q}{c^2D}|C|^{-1/2}\right), \quad (5.13)$$

where

$$a_\infty = \left(\frac{3|C^{(1)}|}{2(1 + c^2D)}\right)^{2/3}, \quad \Theta'_\infty = -\frac{2cD}{1 + c^2D}, \quad Q = g\left|C^{(4)}\right|\frac{Pr - 1}{2Pr}\left(\bar{\lambda}/Pr\right)^{2/3}$$

and $D = C^{(1)}C^{(2)}/4 = 3.448\ldots$ . In these same variables more precise equations (taking into account the contribution of ODLs, see the Appendix, equations (A15), (A16)) for $u = c + \tanh y$ have the form

$$\frac{\mathrm{d}|C|}{\mathrm{d}s} = \frac{2}{3}\frac{a_\infty^{3/2}}{|C|^{1/2}}\left\{1 + \frac{1}{8\pi^{1/2}}\left(1 + \frac{\pi^2c^2}{4}\right)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\int_0^\infty \mathrm{d}x\, x^{-1/2}|C(s-x)|^2 \right.$$
$$\left. + \frac{Q}{|C|^{1/2}}\left[1 + \tfrac{1}{6}(1 + c^2D)\left(\frac{a_\infty^3 Pr}{\pi}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{d}x\, x^{-1/2}|C(s-x)|^{1/2}\right]\right\}, \quad (5.14)$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\bar{\xi}} = \Theta'_\infty\left\{1 + \frac{1}{8\pi^{1/2}}\left(1 - \frac{\pi^2}{4D}\right)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\int_0^\infty \mathrm{d}x\, x^{-1/2}|C(s-x)|^2 \right.$$
$$\left. - \frac{1}{c^2D}\frac{Q}{|C|^{1/2}}\left[1 + \tfrac{1}{6}(1 + c^2D)\left(\frac{a_\infty^3 Pr}{\pi}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{d}x\, x^{-1/2}|C(s-x)|^{1/2}\right]\right\}, \quad (5.15)$$

where $s \equiv \bar{\lambda}\bar{\xi}$. It is evident from (5.13), (5.14) and (5.15) that in this stage the behaviour of $|C(s)|$ and $\Theta'_{\bar{\xi}}(s)$ is determined only by the parameter $Q \sim (Ri/\gamma_L^2)\left(1 - 1/Pr\right)$. The same parameter also determines the role of the stratification.

In the first series of calculations $Pr = 2$, $c = 1$ and $g(\bar{\lambda}/Pr)^{2/3} = 10$, i.e. $Q = \text{const}$. Figures 6 and 7 present the dependences of $\ln|C|$ and $\Theta'_{\bar{\xi}}$ on $\ln s$ for three values of $\bar{\lambda}$: $\bar{\lambda} = 2; 10$ and 250 (respectively, $g = 10$; $2 \times 5^{1/3}$ and 0.4).

Dotted lines correspond to the solution of the asymptotic equations (5.14) and (5.15) with the condition $C(s) = 0$ for $s \leqslant 0$. The dashed line in figure 6 shows the solution of an analogue of equation (5.14) in the unstratified case (G&H). One can

† More precisely, the coordinate $Z$ was used instead of $Y$, $Y(Z) = Y_* \sinh\left(Z/2^{1/2}Y_*\right)$, where $Y_*$ roughly corresponds to a maximum size of the separatrix attained in a particular version of the calculation ($Y_* \approx 2|C|_{\max}^{1/2}$). To the grid uniform in $Z$ there corresponds a grid in $Y$ with the mesh size increasing toward the periphery.

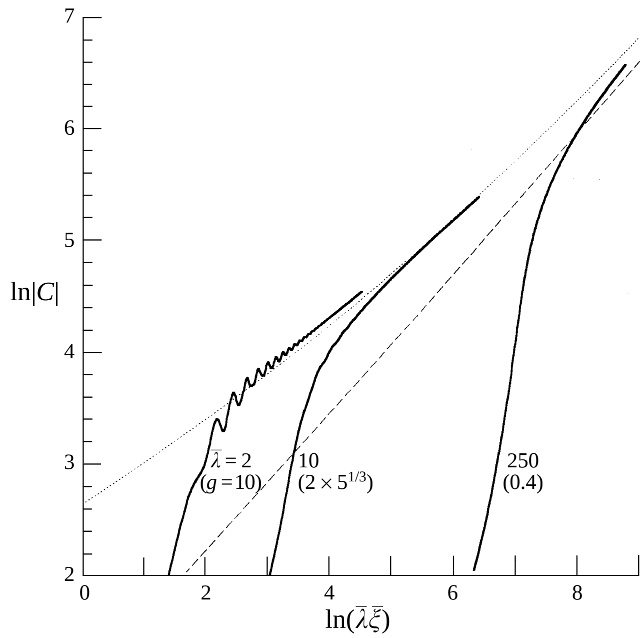


FIGURE 6. $Pr = 2$. Scaled wave amplitude vs. $(\bar{\lambda}\bar{\xi})$ for three values of $\bar{\lambda}$ at a constant $Q$: $g(\bar{\lambda}/Pr)^{2/3} = 10$. Dotted line shows the solution of the asymptotic equation (5.14); dashed line corresponds to the two-term asymptotic expansion for an unstratified flow (from G&H).

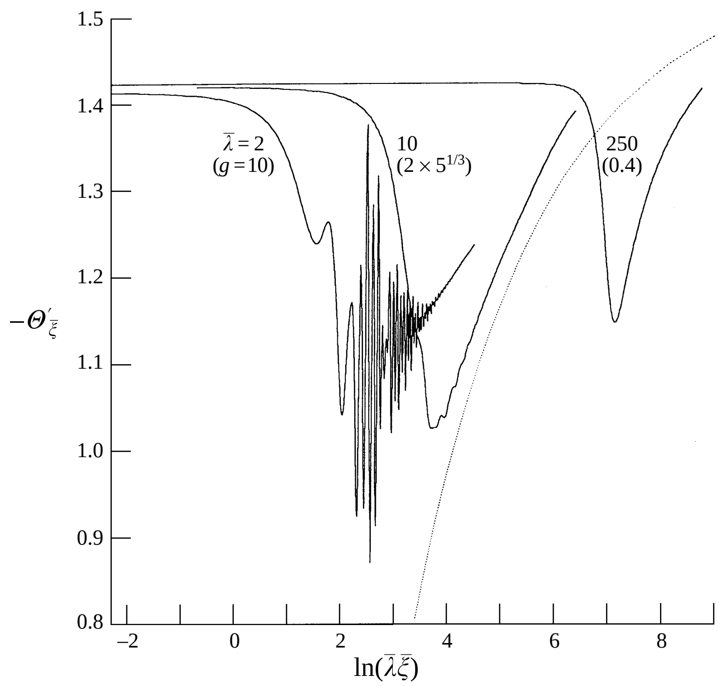


FIGURE 7. Phase change rate $\Theta'_{\bar{\xi}}$ vs. $(\bar{\lambda}\bar{\xi})$ for the same values of parameters as in figure 6. Dotted line shows the asymptotic solution of (5.15).

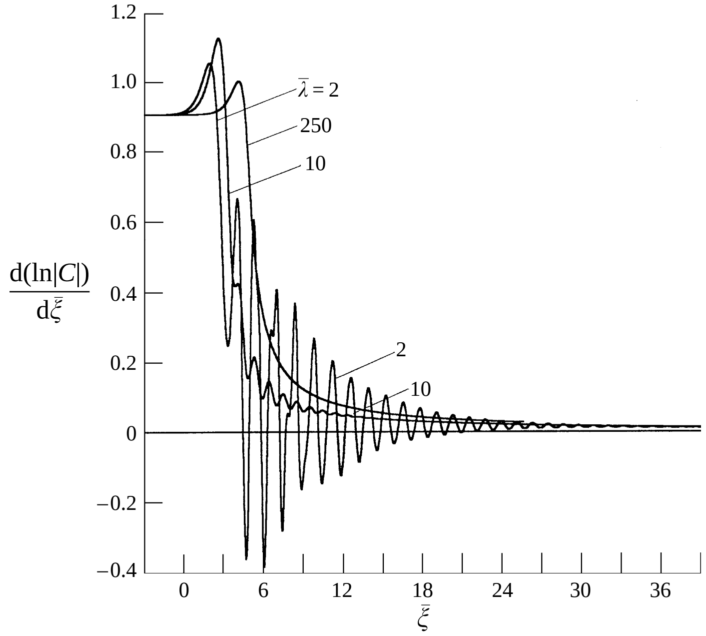


FIGURE 8. Scaled growth rate vs. $\bar{\xi}$ for the same values of parameters as in figures 6 and 7.

see that the asymptotics of the numerical solutions is indeed the same for all $\bar{\lambda}$ and is in good agreement with the analytic solution of the asymptotic equations (5.14) and (5.15).

As in the case of an unstratified flow a monotonic growth of the amplitude is observed at a sufficiently large viscosity (large $\bar{\lambda}$). At the same time there also is one important difference: even at large $\bar{\lambda}$ we cannot describe the entire time history of development by means of a pair of evolution equations (like (5.14) and (5.15)) since when $g \gg 1$ the regions of quasi-steadiness, i.e. the viscous CL ($\gamma_L < \nu^{1/3}$, $A < \nu/Ri^{1/2}$) and the quasi-steady nonlinear CL ($A > \max(Ri, \gamma_L^2)$), are separated by the unsteady CL region and the gap (4.10) (see figure 5) where quasi-steadiness is violated.

At the chosen values of $\bar{\lambda}$ the explosive stage of the disturbance growth in the unsteady CL regime is rather poorly expressed: its presence can be seen only from an increase of the growth rate $|C|^{-1}\mathrm{d}|C|/\mathrm{d}\bar{\xi}$ at the beginning of the post-linear stage of disturbance development (figure 8). Note that in a stratified flow the evolution has a more violent character than in a homogeneous flow for the same values of $\bar{\lambda}$, and the stage of transition into the quasi-steady CL regime is more long-lasting.

These same regularities are also evident in the second series of calculations (figures 9 and 10) where $Pr = 2$ and $g$ is constant ($g = 10$). The constancy of $g$ means the same position of the characteristic points and lines on the amplitude–supercriticality diagram (figures 2 and 5) for all variants of the series: a decrease of the parameter $\bar{\lambda}$ is accompanied only by an increase of $\gamma_L$ (the movement from left to right on the abscissa). Dotted lines in figures 9 and 10 have the same meaning as in figures 6 and 7.

Figure 11 shows the streamwise evolution of the vorticity and density in the critical layer for $Pr = 2$, $g = 10$ and $\bar{\lambda} = 0.1$. The positions of $\bar{\xi}$ shown in figure 11 are

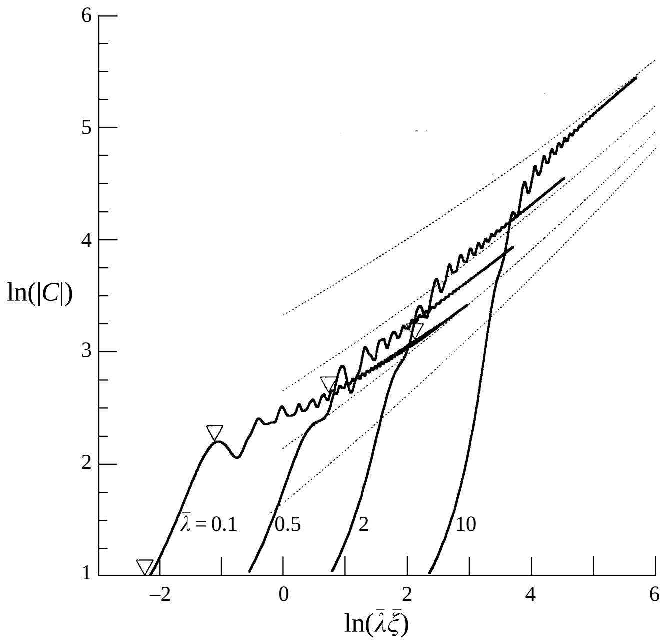


FIGURE 9. $Pr = 2$. Scaled amplitude vs. $(\bar{\lambda}\bar{\xi})$ for several values of $\bar{\lambda}$ at a constant $g = 10$. Dotted lines show the corresponding solutions of (5.14).

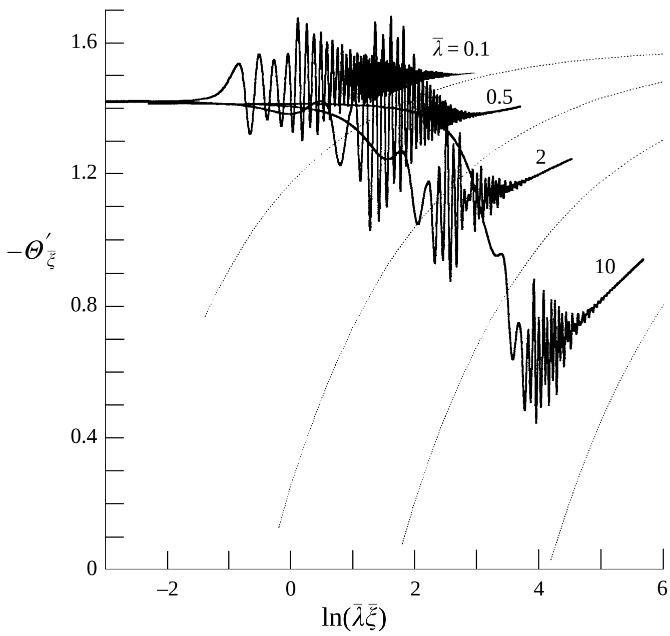


FIGURE 10. Phase change rate $\Theta'_{\bar{\xi}}$ vs. $(\bar{\lambda}\bar{\xi})$ for the same values of parameters as in figure 9. Dotted lines show corresponding solutions of (5.15).

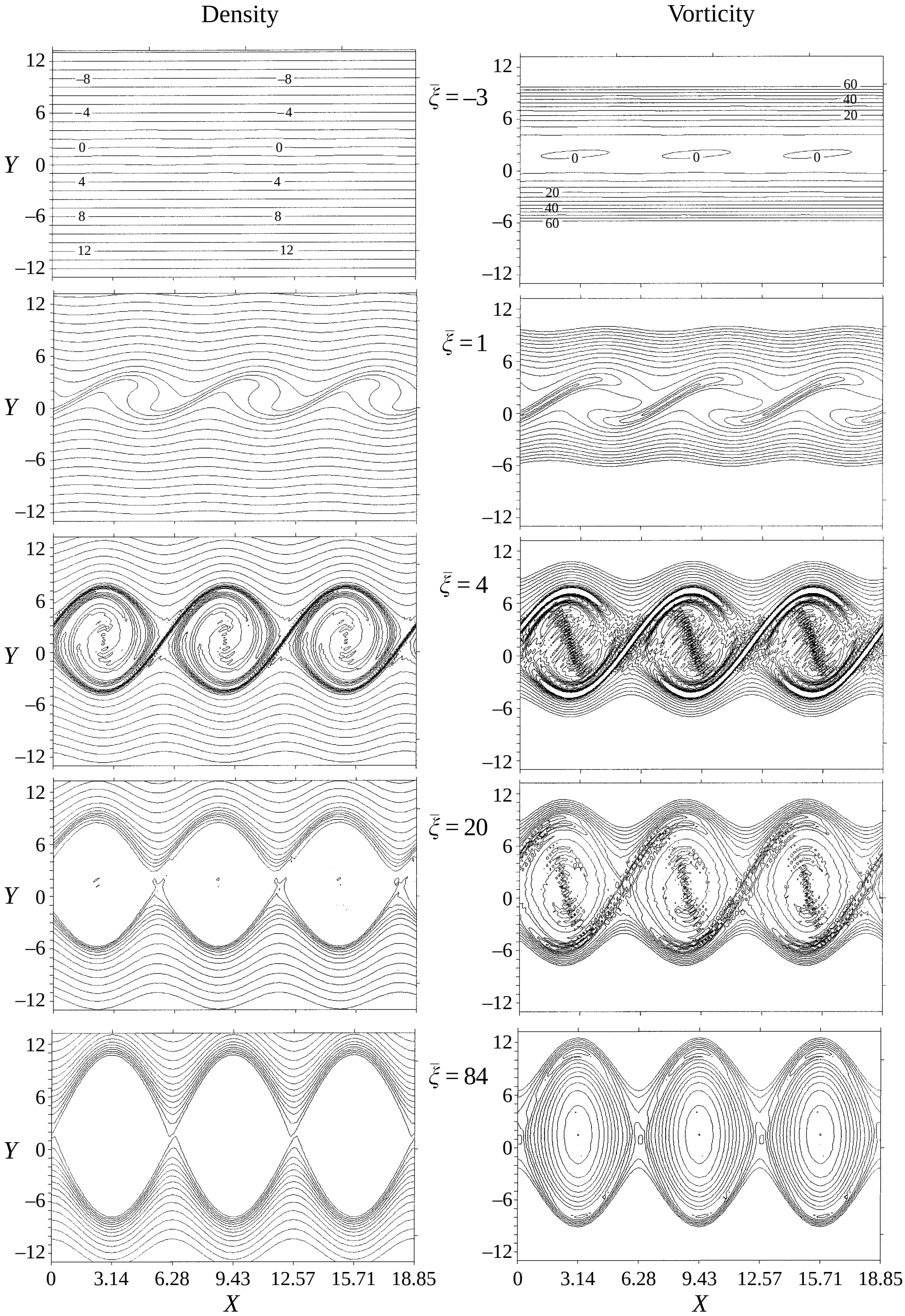


FIGURE 11. Contours of constant vorticity and density on the plane $(X, Y)$ for $\bar{\lambda} = 0.1$, $Pr = 2$ and $g = 10$ for several values of $\bar{\xi}$. These positions are marked in figure 9 by the symbol $\nabla$ .

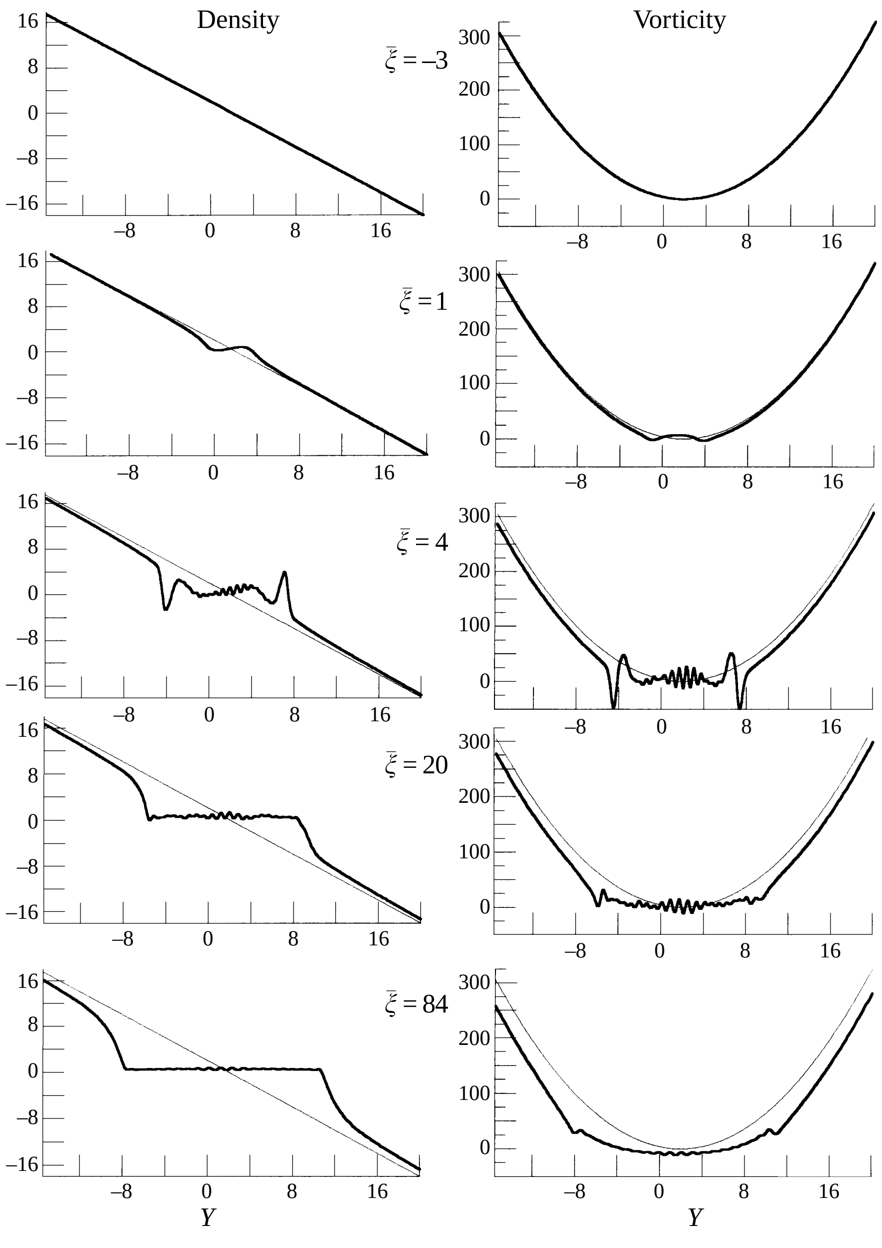


FIGURE 12. Vorticity and density profiles in the central cross-section of the cat's eye $X = \pi$ .

marked on the respective curve of figure 9 by the symbol ∇. More exactly, the isolines

$$-\left(Y - 2/c\right) + P(\bar{\xi}, X, Y) = \text{const}, \quad \left(Y - 2/c\right)^2 + 2\text{Re}\left(C\text{e}^{\text{i}X}\right) - (4/c)\zeta = \text{const} \quad (5.16)$$

are shown, which represent $O\left(\varepsilon^{1/2}\right)$ density and the $O(\varepsilon)$ vorticity in the critical layer. The figures clearly show the formation of a plateau on the density profile and of a smooth distribution of the vorticity inside the cat's eye following a rather violent transition stage (recall that the viscosity is relatively small here, $\bar{\lambda} = 0.1$). It is interesting to note the appearance, at the transition stage, of regions near the cat's

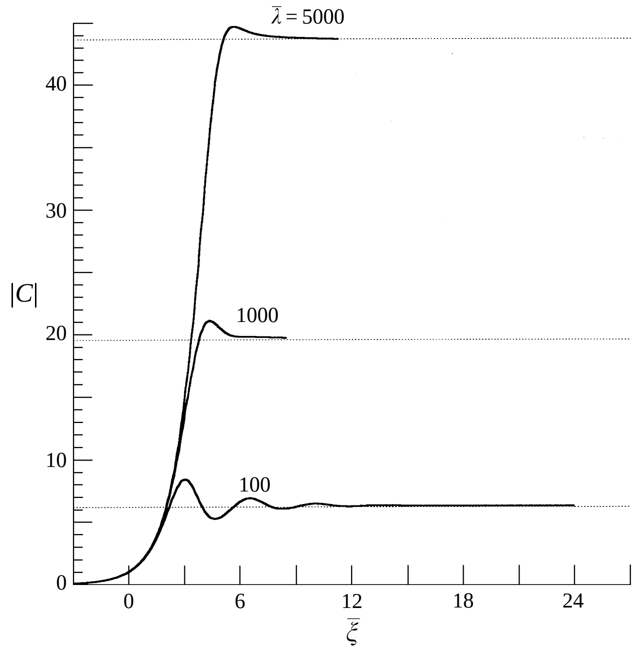


FIGURE 13. Scaled amplitude vs. $\bar{\xi}$ at $Pr = 2/3$, $g = 10$ for several $\bar{\lambda}$ (Scenario I). The dashed line shows the saturation level (5.17).

eye boundaries where the vorticity is significantly smaller than the initial minimum value. Such an effect is impossible in a homogeneous flow, and it is caused entirely by baroclinity. It has been observed also in numerous simulations of stratified shear flows (e.g. Patnaik, Corcos & Sherman 1976; Staquet 1995). This evolutionary feature is more readily illustrated in figure 12, showing profiles of the quantities involved in the left-hand sides of (5.16) in the cross-section running through the cat's eye centre. One can also see clearly a density continuity and a baroclinic jump of the vorticity on the cat's eye boundary, in full conformity with theoretical analysis.

### 5.3. *Results of calculations for* $Pr < 1$

The analysis in §3 has shown that unstable disturbances that start in the viscous CL regime ($\gamma_L < v^{1/3}$, large $\bar{\lambda}$) attain saturation at the level (3.5) (Scenario I), while more unstable disturbances ($\gamma_L > v^{1/3}$, small $\bar{\lambda}$) ultimately reach the nonlinear CL regime. In the latter case, as shown in §4, we have two possible variants of the subsequent evolution.

With a not too large supercriticality ($v^{1/3} < \gamma_L < Ri^{1/2}$), disturbances reach the *unsteady* nonlinear CL regime (the gap (4.10)) and, as shown in §4, cannot go out of it into the region of a quasi-steady nonlinear CL, because the nonlinear growth rate in this region is negative. The evolution in this case (Scenario II) can be determined numerically only .

If, however, $\gamma_L > Ri^{1/2}$, the role of the stratification is negligibly small, and the disturbance from the unsteady CL regime reaches the quasi-steady nonlinear CL regime in the same manner as it does in a homogeneous medium. In this case an exponential growth of the amplitude following some relaxation is replaced by a power-law growth according to a classical law $A \sim x^{2/3}$ (Scenario III).

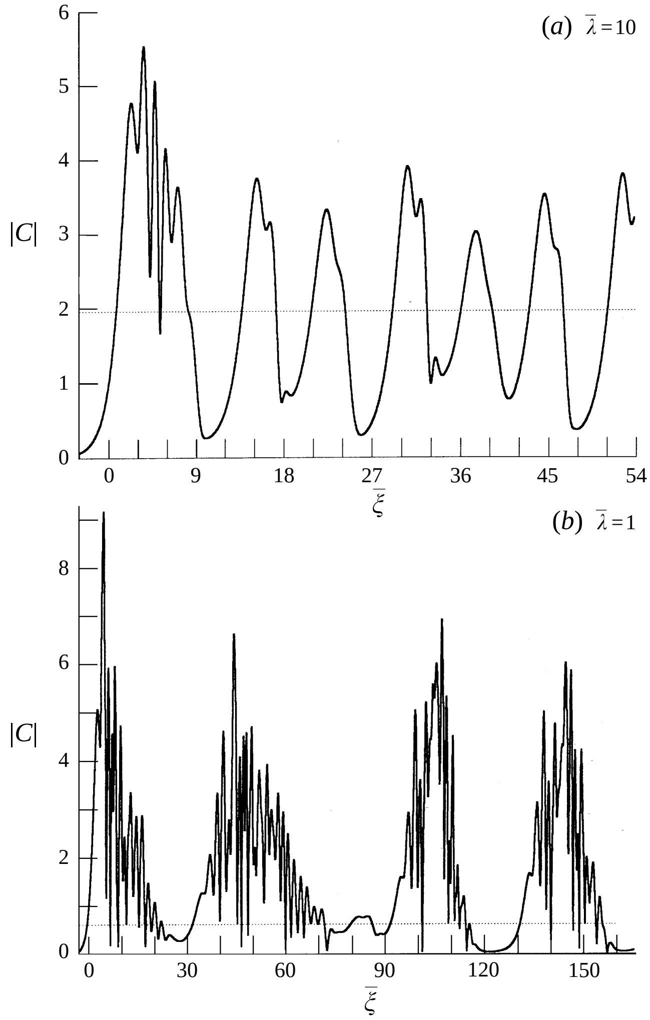


FIGURE 14. As figure 13 but for smaller values of $\bar{\lambda}$ (Scenario II).

Thus, when $Pr < 1$ one must expect the presence of three fundamentally different evolution scenarios alternating with increasing $\gamma_L$ (or decreasing $\bar{\lambda}$). Numerical calculations in general confirm this.

The calculations were carried out for $Pr = 2/3$ and a fixed $g = 10$. Six variants were considered: $\bar{\lambda} = 5000$; 1000; 100; 10; 1; 0.1.

As might be expected, three types of evolution scenarios are evident. Figure 13 presents the dependence of the amplitude $|C|$ on $\bar{\xi}$ for sufficiently large values of $\bar{\lambda}$. One can see that when $\bar{\lambda} \gg 1$ ($\bar{\lambda} = 100$; 1000; 5000) the disturbance does indeed reach saturation in the viscous CL regime (Scenario I). Dashes show saturation levels $|C| = C_{sat}$ that follow from equation (3.3). In the variables (5.1) this level is

$$C_{sat}^2 = \frac{2Pr^{2/3}}{a_4(Pr)(Pr-1)} \frac{\bar{\lambda}}{gc} \tag{5.17}$$

(for $Pr = 2/3$, $a_4(Pr)(Pr-1) \approx 0.40$). It is curious to note that the equilibrium state is approached non-monotonically, which does not agree quite well with the Landau

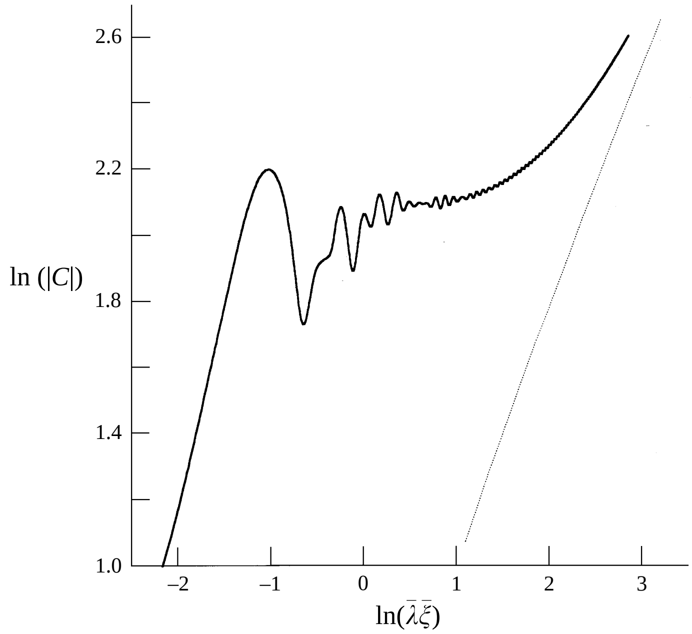


FIGURE 15. As figure 13 but for $\bar{\lambda} = 0.1$ (Scenario III).

equation (3.3) that predicts a monotonic approach of $|C|$ to $C_{sat}$ from below. This difference seems to be attributable to unsteady terms omitted when passing from (3.1) to (3.3). A non-monotonicity of the same kind was discovered by Goldstein & Leib (1989), Leib (1991) and Wu & Cowley (1995) by solving numerically an integro-differential equation of the form (3.1) with a somewhat different kernel for arbitrary values of the parameter $\alpha$ (coincident with $\bar{\lambda}$ provided there is an appropriate scaling).

At smaller $\bar{\lambda}$ (see figure 14) the evolution follows Scenario II: the disturbance enters the *unsteady* nonlinear CL region, the growth of its amplitude is decelerated and then changes to a decrease, and the disturbance returns to the unsteady CL regime where it again begins to grow, etc. This results in something like quasi-periodic oscillations. In any case it can be concluded that neither limitless growth of the amplitude occurs here, as in Scenario I, nor transition into the quasi-steady nonlinear CL regime. The rather early replacement of Scenario I by Scenario II with decreasing $\bar{\lambda}$ is somewhat surprising: it occurs at a relatively small supercriticality (relatively large viscosity), $\bar{\lambda} > 10$, when, at first glance, Scenario I should still occur. This is caused by the same unsteady terms (omitted when deriving (3.3)) which are likely to be insufficiently small.

With a further decrease of $\bar{\lambda}$, we arrive at Scenario III. Figure 15 shows the dependence of $\ln|C|$ on $\ln(\bar{\lambda}\bar{\xi})$ when $\bar{\lambda} = 0.1$.

We observe here, in accordance with the qualitative analysis, the attainment of the asymptotic stage of power-law growth in the regime with a quasi-steady nonlinear CL. On comparing figure 15 with the curve $\bar{\lambda} = 0.1$ in figure 9, one can see that for such values of $\bar{\lambda}$ the stratification is still able to decelerate the evolution compared with the case $Pr > 1$, but it is no longer able to lead to a restriction of the disturbance growth.

To conclude this Section, brief mention should be made of the procedure for choosing computational parameters and of the reliability of results obtained.

The size of the computing semi-interval in $Y$, i.e. the value of $Y_m$, was taken to be $\gtrsim (2\text{–}3)|C|_{max}^{3/4}$, which roughly corresponds to the size of the ODL (in units of (5.1)) at the stage of a developed nonlinear CL. The step in $Y$ was chosen so that at least 5–10 steps were accomodated on a scale of the order of the cat's eye boundary thickness $\Delta_Y \sim \bar{\lambda}^{1/2}/|C|_{max}^{3/4}$. The number of harmonics was varied from 11 to 36 and increased with decreasing $\bar{\lambda}$. The step in $\bar{\xi}$, as has already been mentioned, was chosen automatically. In doing so, the requirement for achieving a preset accuracy led – at some streamwise distances – to a reduction of this step (by dividing it into two) right up to values of $\Delta\bar{\xi} = 0.1 \times 2^{-7} \approx 0.00078$.

The computational reliability was monitored by the reproducibility of results with a change of $Y_m$, the magnitude of the step in $Y$, as well as the number of harmonics. When $Pr > 1$, results of different calculations of the same variant (i.e. for fixed $\bar{\lambda}$ and $g$) are reproduced with confidence right up to the streamwise distances which are shown in the figures and for the presented values of $\bar{\lambda}$ and $g$. For smaller values of $\bar{\lambda}$, $\bar{\lambda} < 0.1$ (and for not too small values of $g$), we were unable to achieve a reliable computation with existing computer resources, hence appropriate results are not presented. When $Pr < 1$ the situation is not so satisfactory. The results presented in figures 14 (*a*) and 15 continue to be reproduced reasonably well with a change of computational parameters. As far as figure 14 (*b*) is concerned, however, its reproducibility is qualitative only. This result should probably be regarded only as an indication that there is no unlimited growth of amplitude.

## 6. Discussion

The above analysis has shown that even a weak stratification can change drastically the nonlinear dynamics of a shear flow instability; it revealed some factors responsible for these changes. When embarking on this work, we were guided, of course, by the experience of previous research (primarily by results reported in G&H, C&S95 and derived by studying the nonlinear evolution in flows with a finite stratification, Churilov & Shukhman 1987*b*, 1988), and based on this experience we tried to predict expected results. It is instructive to consider which of those predictions were borne out and which results obtained turned out to be somewhat surprising.

Stratification makes the neutral mode singular (and a weak stratification makes it weakly singular); therefore, from general considerations (Churilov & Shukhman 1992) it follows that, with a not too weak stratification, there must be the explosive growth stage of disturbances in the unsteady CL regime.

Experience in studying flows with finite stratification suggested that nonlinear properties are determined largely by the competition between diffusion processes of momentum (i.e. viscosity) and density: the nonlinearity will tend to decelerate the development of an instability if the density diffuses faster than the momentum ($Pr < 1$) and accelerate it in the opposite case ($Pr > 1$). This induced us to expect that when $Pr < 1$ there must be instability saturation in the viscous CL regime ($\gamma_L < \nu^{1/3}$) and explosive development with oscillations in the unsteady CL regime, and when $Pr > 1$ explosive growth of the amplitude upon reaching the nonlinearity threshold in each of these two CL regimes.

Finally, based on results of C&S95 it was possible to suppose that, with decreasing stratification parameter (the Richardson number $Ri$) the fast evolution behaviour would be forced out by a slow one, and in two directions at once (see figure 5). In the first place, on the right-hand edge ($\gamma_L$ near 1) of the amplitude–supercriticality diagram a (expanding to the left with a decrease of $Ri$) region of classical slow evolution appears (see G&H) when the exponential growth of the amplitude, following some relaxation of the vorticity distribution inside the CL when $|A| \sim \gamma_L^2$, becomes a power-like one according to the law $|A| \propto x^{2/3}$. Secondly, the explosive growth of the amplitude will be bounded from above by the decreasing (with decreasing $Ri$) level of transition into the nonlinear CL regime. The explosive growth region will ultimately be contracted into a point $|A| \sim \nu^{2/3} \sim \gamma_L^2$ and disappear, and the only way in which the stratification can manifest itself with a further decrease of $Ri$ is a different (from classical) exponent in the law of amplitude growth in some part of the nonlinear CL region. This part will then also contract to a point, and the evolution scenario will become exactly as in G&H. Also, it was fairly obvious that the transition to the nonlinear CL regime means the transition to a slow growth of the amplitude in accordance with a power law, as was the case in all flows studied so far.

The above study has confirmed almost all of these expectations (except for the last one). We have detected both a stabilization in the viscous CL regime when $Pr < 1$ (figure 13) and a stage of explosive growth in the unsteady CL regime, monotonic when $Pr > 1$ and oscillatory when $Pr < 1$ (figures 3 and 4). In numerical calcualtions the explosive stage manifests itself by an excess of the growth rate over $\gamma_L$ in the post-linear stage of development (figure 8). The explosive growth region on the amplitude–supercriticality diagram (figure 5) is indeed bounded on the right ($\gamma_L < Ri^{1/2}$) and from above ($|A| < (\nu Ri)^{2/5}$), becomes narrower with decreasing $Ri$ and contracts into a point when $Ri \sim \nu^{2/3}$. The development of an instability at smaller Richardson numbers was studied earlier (C&S96) and fits quite well into the scheme described above.

It was somewhat surprising to find that, after the transition from the explosive growth stage into the nonlinear CL regime, the flow dynamics inside the CL continues to remain essentially unsteady within a rather broad gap (4.10) of amplitude variation (in figure 5 it is lightly shaded), and only above this gap does the 'habitual' region of quasi-steady nonlinear CL lie (in figure 5 it is shown by dark shading). The asymptotic analysis has shown that when $Pr > 1$ the disturbance can enter this region from below, and when $Pr < 1$ – it cannot. Studying the disturbance evolution inside the gap (4.10), however, required using numerical methods.

The presence of the gap (4.10) has a most dramatic effect on the fate of disturbances when $Pr < 1$. In order for them to grow further, the gap turns out to be an unsurmountable obstacle, with the result that their amplitude performs oscillations in a restricted range ($|A| < Ri$ , figure 14). The range of supercriticalities where a weakly nonlinear restriction of the growth of unstable disturbances occurs is thereby expanding and includes not only the viscous CL region ($\gamma_L < \nu^{1/3}$) but also the explosive growth region in the unsteady CL regime adjacent to it ($\nu^{1/3} < \gamma_L < Ri^{1/2}$).

When $Pr > 1$ disturbances pass sucsessfully through the gap (4.10) and reach the region of the quasi-steady nonlinear CL, so that the course of their evolution (figures 6 and 9) does not differ qualitatively from that described in C&S95.

The existence of the gap (4.10) is a totally new (not encountered earlier) element of the weakly nonlinear evolution scenario, and we have spent much effort in trying to 'close' this gap. Of course, a certain motivation in this case was our reluctance to abandon the idea, confirmed by numerous examples, that in problems of the development of a velocity shear-induced instability, the nonlinear CL is always quasi-steady, and all existing unsteady processes relax sufficiently rapidly on the lower (in amplitude) boundary of this regime. But such efforts were also dictated by a more serious factor.

The comparison of NEEs (3.9) and (4.9) shows their one-to-one correspondence: (4.9) is the result of a term-by-term reduction of (3.9) at the transition into the nonlinear CL regime. On the other hand, if (3.9) is compared with the NEE which Churilov & Shukhman (1988, equation (5.1)) derived for the case of a finite stratification ($Ri \approx 1/4$), we did not detect such a correspondence: it differs greatly from (3.9) by the presence of a *competitive* quintic nonlinear term ($\sim A^5$). It is vital to note that it is this term that ensures a sufficiently low threshold of nonlinearity *throughout the whole region of the unsteady CL* ($\nu^{1/3} < \gamma_L < 1$) and an increase of the growth rate above this threshold, sufficiently fast for sustaining the explosive evolution right to the validity range of weakly nonlinear theory ($|A| = O(1)$) and for preventing the transition into the nonlinear CL regime. Indeed, without the quintic term we would have qualitatively the same evolution pattern as NEE (3.9) provides: the nonlinearity would be competitive only in some part of the unsteady CL region, when $\nu^{1/3} < \gamma_L < \nu^{1/4}$, and on the level $|A| = O(\nu^{3/8})$ the transition into the nonlinear CL regime would occur.

It seemed that the weakly stratified flow problem must have an additional (to that involved in (3.9)) nonlinearity which, in the unsteady CL regime, would be competitive right up to $\gamma_L = O(Ri^{1/2})$ and would raise the level of transition to the nonlinear CL regime to $|A| = O(Ri)$, i.e. to the lower boundary of the quasi-steady nonlinear CL region (see figure 5). A careful analysis showed that there is no other *competitive* nonlinearity † (except for that involved in NEE (3.9)), and hence the gap (4.10) does indeed exist.

The principal reason for its appearance lies in a stronger (compared with C&S95) coupling between the 'singular' and the 'regular' components of the disturbance. Indeed, by comparing equations (2.9) with corresponding equations (2.13) from C&S95, it becomes apparent that while the dynamics of the singular component inside the CL is described by identical equations (2.9*a*) and (2.13*a*), its coupling with the regular component is fundamentally different, but it is the regular component that determines in either case the contribution of the CL to the evolution equations.

In C&S95 this coupling is due solely to unsteady processes (see (2.13*b*) in C&S95), which after the transition into the nonlinear CL regime are forced out rather quickly to the periphery, into the ODLs, and thus coupling becomes significantly weaker. In our problem, however, it is taking place throughout the CL thickness and is sensitive to distribution gradients of the singular component (see (2.9*b*) and especially (2.11)). Because of this, variations in the distribution inside the CL of the singular component (density) cause much more significant (than in C&S95) changes in the distribution of the regular component (vorticity) and, through it, in the dynamics of the amplitude, which in turn has a back influence upon the development of the singular component itself. We now consider qualitatively the mechanism for this feedback.

The transition to the nonlinear CL regime occurs when the amplitude increases so that the unsteady scale becomes of the same order as the nonlinear scale and subsequently smaller than it, i.e.

$$\left|B^{-1}\mathrm{d}B/\mathrm{d}\xi\right| \lesssim B^{1/2} . \tag{6.1}$$

† The problem involves several small parameters ($\gamma_L$, $Ri$, $\nu$) and a full nonlinearity in the evolution equation is representable as an expansion in terms of these parameters and powers of $A$. The largest (in the domain of parameters considered) term out of the terms cubic in $A$ is involved in NEE (3.9). None of the remaining expansion terms can compete with it and with the linear term ($\gamma_L A$) in the range of supercriticalities $\nu^{1/3} < \gamma_L < 1$ below the formal boundary of the nonlinear CL (i.e. when $|A| < \gamma_L^2$).

Physically, the transition means that liquid particles, trapped by the wave, now have sufficient time to move inside the cat's eyes and mix together. In our problem each liquid particle (in the absence of dissipation) has its own value of density, while the vorticity along the trajectory of a liquid particle (unlike the case of a homogeneous medium) is not conserved because of baroclinicity. The mixing is known to be due to non-isochronism of wave-trapped particles motion, i.e. the dissimilar periods of motion in neighbouring orbits. Non-isochronism is especially strong near the separatrix (i.e. the cat's eye boundary), and it is there that refinement of scales of the density distribution starts and proceeds most intensively – while moving in orbits, liquid particles with very differing density values find themselves close together.

There are two processes capable of decelerating the refinement of scales: the diffusion smooths out the density distribution, while a variation in amplitude leads to the most jagged layer near the separatrix either finding itself deep inside the cat's eye (if the amplitude has increased) where non-isochronism is much weaker, or (if the amplitude has decreased) being are forced out of the cat's eye, into the region of transit particles. We will now proceed to show that in our problem the refinement of scales comes to a halt just because of a variation in amplitude and on a scale larger than the scale at which diffusion comes into play.

Difusive smoothing that leads to the formation of a quasi-steady nonlinear CL starts when the dissipation term in the operator $\tilde{\mathscr{L}}$ (see (2.8) and below) becomes of the same order as the other terms, i.e. at the scale of refinement $L \sim L_D = \eta/B$. At larger scales, there is still no smoothing, and the nonlinear CL cannot become quasi-steady. As is evident from (2.11), the function $G$ (i.e. the part of the vorticity which influences the amplitude evolution) increases rapidly with decreasing refinement scale of the density $L$:

$$G \sim \frac{\eta J P}{L^2 B} \sim \frac{\eta J}{L^2 B^{1/2}} ,$$

because, according to (4.4), $P = O\left(B^{1/2}\right)$ (this relationship is true not only in the quasi-steady limit but also in the general case). Corresponding contributions to the right-hand sides of MSCs (2.10) increase together with $G$. It is easy to obtain an estimate of the amplitude variation rate:

$$\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}\xi} \sim \frac{GL}{B} \sim \frac{\eta J}{LB^{3/2}} \sim \frac{L_D}{L}\frac{J}{B^{1/2}} .$$

Under the condition $B < J$ we find that

$$\left|\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}\xi}\right| > \frac{L_D}{L}B^{1/2} \quad \text{or} \quad l_t > \frac{L_D}{L}\, l_N.$$

Since in the nonlinear CL regime the inequality (6.1) is satisfied, i.e. $l_N > l_t$, we conclude that, as long as $B < J$ (i.e. $|A| < Ri$), $L$ remains larger than $L_D$. This does indeed mean that the scale of refinement cannot become so small that a smoothing of the density distribution occurs, which opens the way to the quasi-steady regime of the nonlinear CL, and the disturbance evolution in the nonlinear CL regime will be unsteady right up to the amplitudes $A \sim Ri$†.

A relationship between the refinement of the density distribution and the vorticity

† Since at $Pr < 1$ this level cannot be reached, these sharp (but insufficiently sharp) gradients cannot be smoothed away by diffusive processes. Recently Staquet (1995) has shown by numerical simulations that in the regions of the most sharp gradients which are localized near the cat's eyes boundaries (baroclinic layers in author's terminology) secondary instabilities develop and turbulence arises.

oscillations is clearly seen in figures 11 and 12 showing the appearance, in the course of the evolution, of regions where the vorticity takes large negative values, while the vorticity of an undisturbed flow (the top plot in figure 12) is everywhere non-negative – in a homogeneous flow such an effect is impossible. This relationship manifests itself even in the quasi-steady case ($|A| \gg Ri$) in the form of a baroclinic jump of the vorticity on the cat's eye boundary, i.e. just where the density gradient changes abruptly (see C&S96, Appendix A).

It should be noted that an analogy between the density distribution in a weakly stratified flow and the vorticity in a homogeneous flow was first established by Haberman (1973). On its basis he considered a steady nonlinear CL in a weakly stratified medium and showed that the density, which is constant inside the cat's eyes, in accordance with Grimshaw's (1969) results, must be continuous on their boundaries in the limit of a vanishingly small dissipation and must have quite a definite gradient jump on the cat's eye boundaries and quite a definite jump across the CL. In this study we have established that always when $Pr > 1$, and when $Pr < 1$ in the case of a sufficiently large supercriticality ($\gamma_L > Ri^{1/2}$), as a result of instability development, a flow structure inside the CL forms, which coincides not only qualitatively but also quantitatively with that predicted by Haberman. Based on this we provided one of the possible answers to the "question... of determining the evolution mechanisms for generation of such finite-amplitude modes" which was posed by Haberman (1973).

We wish to note in conclusion that the 'dissimilarity' of the evolution equations (3.9) for weakly stratified flow and the NEE for the flow with finite stratification (Churilov & Shukhman 1988) raises the question of the continuity of the transition from a finite stratification to its absence through a weak stratification. Because the only case of a finite stratification ($Ri \approx 1/4$) that has been considered so far is the limiting one in certain sense (there is no instability at a larger $Ri$), to answer this question requires a theory for nonlinear instability development of a shear flow when $0 < Ri < 1/4$.

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## Appendix. Corrections for unsteadiness to the nonlinear evolution equation in the quasi-steady nonlinear CL regime

Unsteady terms involved in equations (2.9*a,c*) and (2.11) inside the quasi-steady nonlinear CL ($Y - Y_c = O(1)$) are small compared not only with nonlinear but also with dissipation terms and become of the same order as the latter only within the outer diffusive layers (ODL) $\left(Y - Y_c = O\left((\eta/\Gamma)^{1/2}\right)\right)$, which connect the CL with outer flow regions. Through solutions in them matching of the inner (in the CL) solution to the inner asymptotics (2.5) of the outer solution is accomplished. To render these facts clearer, it is convenient to introduce the parameter $\mu \ll 1$ ($\Gamma = O(\mu)$) and the variables $t = \mu\tau = \mu\xi/c,\ s = (\mu/\eta)^{1/2}(Y - Y_c)$, as well as, in addition, to separate the sine ($F$) and cosine ($H$) parts of $\zeta_h$ generated, respectively, by terms with $\sin\theta$ and $\cos\theta$ on the right-hand side of (2.9*c*). Equations (2.9*a,c*) and (2.11) in the ODL take the form:

$$ku_c' sP_\theta = -\frac{\mu}{\eta}2kB\sin\theta P_s + \left(\frac{\mu^3}{\eta}\right)^{1/2}\left(\frac{1}{Pr}P_{ss} - P_t\right) + \left(\frac{\mu}{\eta}\right)^{1/2} 2kB\sin\theta + \cdots\,, \quad \text{(A 1)}$$

$$ku_c' sG_\theta = -\frac{\mu}{\eta}2kB\sin\theta G_s + \left(\frac{\mu^3}{\eta}\right)^{1/2}(G_{ss} - G_t) - \frac{\mu^2}{\eta}\frac{Pr-1}{Pr}Ju_c' P_{sss} + \cdots\,, \quad \text{(A 2)}$$

$$ku_c' sF_\theta = -\frac{\mu}{\eta}2kB\sin\theta F_s + \left(\frac{\mu^3}{\eta}\right)^{1/2}(F_{ss} - F_t) - \left(\frac{\mu}{\eta}\right)^{1/2}\frac{2u_c'''}{u_c'}(\Omega - \dot{\Theta})B\sin\theta + \cdots\,, \quad \text{(A 3)}$$

$$ku_c' sH_\theta = -\frac{\mu}{\eta}2kB\sin\theta H_s + \left(\frac{\mu^3}{\eta}\right)^{1/2}(H_{ss} - H_t) - \left(\frac{\mu^3}{\eta}\right)^{1/2}\frac{2u_c'''}{u_c'}\frac{\mathrm{d}B}{\mathrm{d}t}\cos\theta + \cdots\,, \quad \text{(A 4)}$$

where $\zeta_h = F + H$, $\dot{\Theta} = \Theta_\tau$†. The solution inside the CL is defined, with the unsteadiness neglected, by equations (4.2)–(4.4). We start the analysis with unstratified contributions $F$ and $H$. We construct the solution inside the CL as an expansion of the form

$$F = F_0 + (\mu/\eta)^{1/2} F_1 + \cdots, \qquad H = H_0 + \cdots\,,$$

while in the ODL it is more convenient to expand in terms of the harmonics $\theta$:

$$F = F^{(0)} + \cdots, \qquad H = H^{(0)} + H_S^{(1)}\sin\theta + H_C^{(1)}\cos\theta + \cdots\,.$$

According to (4.4),

$$F_0 = \frac{u_c'''(\Omega - \dot{\Theta})}{k\left(2Bu_c'^3\right)^{1/2}} Bg_1(\lambda; z, \theta)\,.$$

When $|z| \to \infty$, as shown by Haberman (1972), $g_1 = \sigma C^{(1)}/2 + O\left(\cos\theta/z\right)$, $\sigma = \mathrm{sign}(z)$, so that

$$F_0 = \frac{\sigma u_c''' C^{(1)}}{2k\left(2Bu_c'^3\right)^{1/2}}(\Omega - \dot{\Theta})B + O\left(\frac{\cos\theta}{Y}\right)\,.$$

The first term is independent of $\theta$ and represents the well-known vorticity jump across the CL, because of which the inner solution does not match to the asymptotic expansion (2.5) of the outer solution straightforwardly. Appropriate matching is done through the zeroth harmonic $F^{(0)}$ of the solution in the ODL and is similar to that performed in the C&S95. At $O(1)$ function $F^{(0)}$ for $s > 0$ and $s < 0$ (the + and − signs, respectively) is determined from the solution of the boundary-value problem

$$\left.\begin{aligned} &F_{ss}^{(0)\pm} - F_t^{(0)\pm} = 0, \\ &F^{(0)\pm}(0,t) = \pm\frac{u_c''' C^{(1)}}{2k\left(2Bu_c'^3\right)^{1/2}}(\Omega - \dot{\Theta})B; \qquad F^{(0)\pm} \to 0 \quad \text{as} \quad s \to \pm\infty. \end{aligned}\right\} \quad \text{(A 5)}$$

This solution

$$F^{(0)\pm}(s,t) = \frac{u_c''' C^{(1)}}{8k\left(\pi u_c'^3\right)^{1/2}} s\int_0^\infty \mathrm{d}t_1\, t_1^{-3/2}\left[2B(t-t_1)\right]^{1/2}\left[\Omega - \dot{\Theta}(t-t_1)\right]\exp\left(-\frac{s^2}{4t_1}\right)$$

† As is evident from (4.8*b*), even in the quasi-steady limit $\Theta_\tau = O(1)$ (but $\Theta_{\tau\tau} = O(\mu)$ !); therefore, using $\Theta_t = \mu^{-1}\Theta_\tau$ may provide a misleading perception of the order of a corresponding term.

when $s \to \pm 0$ has the asymptotic expansion

$$F^{(0)\pm}(s,t) = \pm\frac{u_c''' C^{(1)}}{4k}\left[\frac{2B(t)}{u_c'^3}\right]^{1/2}\left(\Omega - \dot{\Theta}(t)\right) - \frac{u_c''' C^{(1)}}{4k\left(\pi u_c'^3\right)^{1/2}} s\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mathrm{d}t_1\, t_1^{-1/2}\,[2B(t-t_1)]^{1/2}\left(\Omega - \dot{\Theta}(t-t_1)\right),$$

and to match with it the contribution $F_1$ at $O\left((\mu/\eta)^{1/2}\right)$ of the inner solution must have following the asymptotic behaviour when $|z| \to \infty$:

$$F_1 = 2D_1(t)z, \quad D_1 = -\frac{u_c''' C^{(1)}}{8ku_c'^2}\left[\frac{2B(t)}{\pi}\right]^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mathrm{d}t_1\, t_1^{-1/2}\,[2B(t-t_1)]^{1/2}\left(\Omega - \dot{\Theta}(t-t_1)\right). \tag{A 6}$$

It is easy to see that $F_1$ satisfies the equation $\mathscr{M}F_1 = 0$, the solution of which with the required asymptotic behaviour is $F_1 = D_1\tilde{g}_1$, $\tilde{g}_1 = g_1 + 2z$. If we write $F_0 = D_0(t)(\tilde{g}_1 - 2z)$, one can then see that $F_1$ and subsequent iterations obtained in the same way give only a renormalization of the coefficient at $\tilde{g}_1$.

In this manner, with an accuracy up to $O\left((\mu/\eta)^{1/2}\right)$,

$$\left.\begin{aligned} \fint_{-\infty}^{\infty}\mathrm{d}Y\,\langle F\sin\theta\rangle = \frac{\eta u_c''' C^{(1)}}{k^2\left(8Bu_c'^3\right)^{1/2}}\Bigg\{\Omega - \dot{\Theta} - \frac{C^{(1)}}{4}\left(\frac{\mu}{\pi\eta u_c'}\right)^{1/2} \\ \times\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mathrm{d}t_1\, t_1^{-1/2}\,[2B(t-t_1)]^{1/2}\left(\Omega - \dot{\Theta}(t-t_1)\right)\Bigg\}, \quad \fint_{-\infty}^{\infty}\mathrm{d}Y\,\langle F\cos\theta\rangle = 0. \end{aligned}\right\} \tag{A 7}$$

It can be shown that the $\cos\theta$ contribution appears not in the next order, $O(\mu/\eta)$, but much later (see C&S95).

The right-hand side of equation (4.6$a$) with a correction for unsteadiness has thereby been calculated.

Further, in accordance with (4.4), the cosine term inside the CL in the first approximation is

$$H_0 = \frac{\mu u_c''' B_t}{k\left(2Bu_c'^3\right)^{1/2}}\, g_2(\lambda; z, \theta).$$

When

$$|z| \to \infty \quad g_2 = -\frac{1}{\lambda}\ln|z| + g_{20} - \frac{2\sin\theta}{z} - \frac{\cos\theta}{\lambda z^2} + O\left(z^{-3}\right),$$

so that

$$H_0 = -\frac{\mu}{\eta}\frac{u_c'''}{u_c'^2}\left[\frac{\mathrm{d}B^2}{\mathrm{d}t}\ln|s| + \frac{2(\mu\eta)^{1/2}}{k}\frac{\mathrm{d}B}{\mathrm{d}t}\frac{\sin\theta}{s} + \frac{4\mu}{\eta}\frac{B^2}{u_c'}\frac{\mathrm{d}B}{\mathrm{d}t}\frac{\cos\theta}{s^2} + \cdots\right] + H_{00}(t). \tag{A 8}$$

A constant $H_{00}$ (and, together with it, also $g_{20}$) is determined from matching to the outer solution.

As in the case of the sine contribution $F$, $\theta$-independent terms of the asymptotic representation (A8) must be reduced to zero in the ODL. Consider equation (A4). The main contribution to the fundamental harmonic is of $O\left(\mu^{3/2}/\eta^{1/2}\right)$. It is obtained by a direct integration:

$$H_S^{(1)} = -\left(\frac{\mu^3}{\eta}\right)^{1/2}\frac{2u_c'''}{ku_c'^2}\frac{\mathrm{d}B}{\mathrm{d}t}\frac{1}{s} \tag{A 9}$$

and is matched both to the second term in (A8) and to (2.5). The first term on the right-hand side of (A4) plays the role of a 'generator of harmonics'. Substitution of

(A9) into it gives contributions to the zeroth and second harmonics at $O(\mu/\eta)$ and $O\left(\mu^{5/2}/\eta^{3/2}\right)$, respectively.

The contribution to the zeroth harmonic is described by

$$\left(\frac{\partial^2}{\partial s^2}-\frac{\partial}{\partial t}\right)H^{(0)}=\frac{\mu}{\eta}\frac{u_c'''}{u_c'^2}\frac{\mathrm{d}B^2}{\mathrm{d}t}\frac{1}{s^2},\qquad H^{(0)}\to 0\quad \text{as}\quad s\to\pm\infty\,. \tag{A 10}$$

It is evident that when $s\to 0$ the asymptotic behaviour of its solution is singular and contains the same logarithm as the right-hand side of (A8). At the same time the solution (A10) is defined so far up to an arbitrary solution of the homogeneous equation tending to zero when $|s|\to\infty$. The constraint

$$\frac{\partial}{\partial s}\left(H^{(0)}+\frac{\mu}{\eta}\frac{u_c'''}{u_c'^2}\frac{\mathrm{d}B^2}{\mathrm{d}t}\ln|s|\right)\to 0\qquad \text{as}\quad s\to 0 \tag{A 11}$$

is to be added as the missing boundary condition. Indeed, it is easy to see that $H^{(0)}(-s)=H^{(0)}(s)$ and if this limit is non-zero, then the inner ($s\to 0$) asymptotic expansions of solutions in ODLs will contain the term $\sim|s|$ which is impossible to match to the solution inside the CL (see C&S95, p. 70). The solution of (A10), (A11) has the form

$$H^{(0)}=\frac{\mu}{\eta}\frac{u_c'''}{u_c'^2}\frac{\partial U}{\partial s},\qquad U(s,t)=\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\frac{\mathrm{d}t_1}{t_1}\,[B(t-t_1)]^2\,\Phi\left(1,\tfrac{3}{2};-\frac{s^2}{4t_1}\right), \tag{A 12}$$

where $\Phi(a,b;x)$ is Kummer's confluent hypergeometric function. When $s\to 0$

$$H^{(0)}(s,t)=-\frac{\mu}{\eta}\frac{u_c'''}{u_c'^2}\left[\frac{\mathrm{d}B^2}{\mathrm{d}t}\left(\ln|s|+\tfrac{1}{2}C_E\right)-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^\infty\!\mathrm{d}t_1\,[B(t-t_1)]^2\ln t_1+O\left(s^2\ln s\right)\right],$$

i.e.

$$H_{00}(t)=\frac{\mu}{2\eta}\frac{u_c'''}{u_c'^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^\infty\mathrm{d}t_1\,[B(t-t_1)]^2\ln t_1-C_E\right]\,.$$

Here $C_E=0.577216\ldots$ is Euler's constant.

On substituting (A12) into (A4) and integrating over $\theta$, we obtain, using (A10), the contribution to the fundamental harmonic at $O\left(\mu^2/\eta^2\right)$,

$$H_{\mathrm{C}}^{(1)}=\left(\frac{\mu}{\eta}\right)^2\frac{2Bu_c'''}{u_c'^3}\left(\frac{1}{s}\frac{\partial U}{\partial t}-\frac{1}{s^2}\frac{\mathrm{d}B^2}{\mathrm{d}t}\right), \tag{A 13}$$

which, judging from its structure, gives a correction for unsteadiness to the right-hand side of the (4.6*b*).

The last term on the right-hand side of (A13) is a continuation into the ODL of the last term in square brackets in (A8), i.e. its contribution to (4.6*b*) is already included. The first term on the right-hand side of (A13), however, should be matched to the not yet calculated iteration of the solution inside the CL having the same order, $O\left(\mu^2/\eta^2\right)$. The integral over the inner region appearing on the left in (4.6*b*) consists of the integral over CL and the integrals over two ODLs sandwiching it. Since the order of magnitude of the integrand in the CL and ODLs is the same and the ODL scale far exceeds the CL scale, the main contribution to the integral will be made by ODLs. Thus, the correction for unsteadiness to (4.6*b*) due to the first term on the right-hand side of (A13) is

$$\delta\left(\,\rlap{-}\!\!\int_{-\infty}^{\infty}\mathrm{d}Y\,\langle H\cos\theta\rangle\right)=\left(\frac{\pi\mu}{\eta}\right)^{3/2}\frac{u_c'''}{2u_c'^3}\,B\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^\infty\!\mathrm{d}t_1\,t_1^{-1/2}[B(t-t_1)]^2+O\left(\frac{\mu^2}{\eta^2}\right), \tag{A 14}$$

whereas $\int\mathrm{d}Y\,\langle H\sin\theta\rangle=0$.

Finally, the stratification-driven contribution is determined from the combined solution of equations (A1), (A2) in ODLs and (2.9*a*), (2.11) in the CL. The calculation of $P$ is the same as that of $F$. As a result, we find that taking into account the corrections for unsteadiness reduces to a renormalization:

$$P=-\left(\frac{B}{2u_c'}\right)^{1/2}\left\{g_1\left(\frac{\lambda}{Pr};z,\theta\right)-\frac{C^{(1)}}{4}\left(\frac{\mu Pr}{\pi\eta u_c'}\right)^{1/2}\tilde{g}_1\left(\frac{\lambda}{Pr};z,\theta\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mathrm{d}t_1 t_1^{-1/2}[2B(t-t_1)]^{1/2}\right\}$$

and, consequently,

$$G=\tfrac{1}{2}Ju_c'\tilde{g}_4(\lambda,Pr;z,\theta)\left\{1-\frac{C^{(1)}}{4}\left(\frac{\mu Pr}{\pi\eta u_c'}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mathrm{d}t_1\, t_1^{-1/2}[2B(t-t_1)]^{1/2}\right\},$$

$$\int_{-\infty}^{\infty}\mathrm{d}Y\,\langle G\cos\theta\rangle=\frac{Pr-1}{2Pr}J\left(2Bu_c'\right)^{1/2}C^{(4)}\left\{1-\frac{C^{(1)}}{4}\left(\frac{\mu Pr}{\pi\eta u_c'}\right)^{1/2}\right.$$
$$\left.\times\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mathrm{d}t_1\, t_1^{-1/2}[2B(t-t_1)]^{1/2}\right\},\qquad \int_{-\infty}^{\infty}\mathrm{d}Y\,\langle G\sin\theta\rangle=0\,.$$

(Here $\tilde{g}_4=g_4-\partial g_1\left(\lambda/Pr;z,\theta\right)/\partial z$).

All desired corrections are thereby calculated. Taking them into account we obtain for the nonlinear evolution equations (4.8*a,b*) the more precise forms

$$\frac{\mathrm{d}B}{\mathrm{d}\xi}=-\frac{\eta C^{(1)}}{\Delta(0)u_c'}\left\{\frac{ku_c'''I_2}{\left(2Bu_c'\right)^{1/2}}\Omega+\frac{\Delta(\infty)}{4}\left(\frac{cu_c'}{2\pi\eta^3B}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\int_0^\infty \mathrm{d}\xi_1\,\xi_1^{-1/2}[B(\xi-\xi_1)]^2\right.$$
$$\left.+\frac{Pr-1}{4Pr}\frac{cu_c'''C^{(4)}}{B}J\left[1-\frac{C^{(1)}}{4}\left(\frac{cPr}{\pi\eta u_c'}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^\infty \mathrm{d}\xi_1\,\xi_1^{-1/2}[2B(\xi-\xi_1)]^{1/2}\right]\right\},\qquad (\mathrm{A}\,15)$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}=\left(1+\frac{2k^2I_0I_2}{\Delta(0)}\right)\frac{\Omega}{c}-\left(\frac{c^3}{\pi\eta^3}\right)^{1/2}\frac{I_0ku_c'''}{2\Delta(0)u_c'^{\,3}}\left(C^{(1)}C^{(2)}-\pi^2\right)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\int_0^\infty \mathrm{d}\xi_1\,\xi_1^{-1/2}[B(\xi-\xi_1)]^2$$
$$+\frac{Pr-1}{Pr}\left(\frac{u_c'}{2B}\right)^{1/2}\frac{kI_0}{\Delta(0)}C^{(4)}J\left[1-\frac{C^{(1)}}{4}\left(\frac{cPr}{\pi\eta u_c'}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^\infty \mathrm{d}\xi_1\,\xi_1^{-1/2}[2B(\xi-\xi_1)]^{1/2}\right].\qquad (\mathrm{A}\,16)$$

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